

THE ANALYSIS OF TRAVELLING WAVES ON  
POWER SYSTEM TRANSMISSION LINES

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by

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# CORRIGENDUM

Page 33 Second paragraph first sentence: equation (3.3) ... equation (3.2).

Page 38 equations ((3.2) and (3.3) ....

Page 73 Fig. (7.1):  $\frac{dt}{dx} = \dots$

Page 77 Eqn (7.12):  $\frac{\lambda_2}{\lambda_1} = \sqrt{\frac{L}{C}}$  ,  $\frac{\lambda_2}{\lambda_1} = -\sqrt{\frac{L}{C}}$

Page 78 Eqn (7.13):  $R_i$  should read  $R_i$

Page 93  $\gamma = \dots \frac{RG}{S^2_{LC}})^{\frac{1}{2}}$

Page 94 Final eqn ...  $-\sigma e^{-\frac{\rho x}{a}} \int \dots$

Page 116 First paragraph last sentence: 10.2.2) part (d)

Fig. (10.1): Length  $\Delta l$ .

Page 119 Part (b) first sentence:

$$i(x,t) = i(x_1, t_1) + \frac{i(x_s, t_s) - i(x_1, t_1)}{\Delta x_1} (x - x_1)$$

for  $x_s \geq x \geq x_1$  on curve  $C_1$ .

Page 127 First sentence: p.u. system.

Page 141 Last equation:  $= \begin{Bmatrix} 0 \\ -L_s \\ -L_m \end{Bmatrix}$

Page 143 Eqn (11.9): Interchange R and G.

$$\text{Eqn (11.10): } \left[ \begin{matrix} R_s - R_m \\ G_s + G_m \end{matrix} \right]^{-1} \quad \text{Eqn (11.11): } \left[ \begin{matrix} R_s + R_m \\ G_s + G_m \end{matrix} \right]^{-1}$$

Page 167 Denominator in eqn  $\Delta V_{c1} = : (\frac{1}{Z1} + \frac{1}{Z2})$

Page 209 Part (b) first eqn:  $[\frac{(1-t)}{2} + \frac{t(t+x)}{8}]$

Page 221 c speed of light

Page 241 Integrand denominator  $P = : \sqrt{T^2 + 1} + T$

Page 252 Last sentence: exponential mode

Page 304  $T_i(t_0) \rightarrow 0$  as  $t_0 \rightarrow T_0$  where  $T_0$  is very small.

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ABSTRACT

This thesis is concerned with the mathematical representation of, and the analysis of travelling waves on, power system transmission lines.

The transmission lines are described mathematically by the Telegrapher's equations. Two variants are considered in detail; where the equation coefficients are constant and where they are functions of frequency. A third variant where the coefficients are functions of the line voltage is introduced but not given a detailed treatment.

The travelling wave analysis consists of finding solutions of the Telegrapher's equations, bearing in mind the characteristics of the transmission lines, the equipment to which they will be connected and the nature of the signals likely to be experienced in power system transient situations.

The constant coefficient Telegrapher's equations form the basis of the 'Linear Transmission Line Analysis', the first part of this thesis. The assumptions involved in this representation and the approximations that they introduce are considered. The many techniques available for solution are critically reviewed. Even for this, the simplest travelling wave representation of a power system transmission line, solution of transient problems can introduce considerable difficulties. Application of the method of characteristics to the solution of transient propagation problems is studied in detail. This method of solution is shown to have a much wider application than has generally been recognised. Its

application has led to a number of new solutions which are shown to be superior to those in current use. Both single circuit and mutually coupled multiple circuit transmission lines have been considered. In order to facilitate possible future extension of this work into the area of non-linear transmission special consideration has been given to the direct application of time domain solution methods to transient propagation problems.

The Telegrapher's equations with frequency dependent coefficients form the second part of this thesis. Literature in this subject area is both copious and scattered making it a time-consuming process to attain a current knowledge of this work. To alleviate this situation a critical summary and discussion of significant results in both the formulation of these effects and solution of the resulting equations is given. Presentation is detailed where necessary. In a number of instances the work being discussed is developed beyond that originally given.

A brief concluding section considers the current state of development towards a comprehensive first order power system transmission line representation.

ACKNOWLEDGEMENTS

I wish to express my gratitude to my supervisor, Mr W.H. Bowen, for his encouragement and support during the course of this project.

I am indebted to other members of the Electrical Engineering Department for many discussions on aspects of this work. In particular I would like to thank Dr J.H. Andreae.

I would like also to thank Dr W.B. Wilson of the Mathematics Department for his helpful discussions on aspects of complex variable analysis.

During the course of this project I was the grateful recipient of a University Grants Committee Scholarship.

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## PREFACE

The motivation for the work contained in this thesis was an obsession with the Telegrapher's Equations and in particular their solution by the method of characteristics. This should be borne in mind when surveying the rather unconventional structure of this thesis. From this motivation grew a more broad interest in the propagation of transient signals throughout power systems which finally crystallized itself in a desire to work towards obtaining a workable representation of a power system transmission line that could be used to study transient propagation and which would contain all the first order propagation effects experienced by a signal propagating along such a line.

There are sound reasons why such an analysis of transient propagation is of benefit to engineering science. The ability to describe propagation of transient signals throughout power system networks is becoming increasingly important as system voltages rise and system complexity increases. Protection, insulation coordination and other overvoltage related problems all benefit from a better understanding and description of travelling waves on transmission lines.

Broadly speaking there are three first order effects to be considered. The first, and most important of these, is the property of propagation. This is formulated by the Wave Equation which for a power system transmission line is described in its most rudimentary form by the Transmission Line Equations [T.L.E.s] (previously and often referred to as

the Telegrapher's Equations). In the simplest description of propagation the coefficients of the T.L.E.s are assumed constant. However, in a typical power system transient situation this assumption is seldom true and depending on the requirements of the solution being sought, the coefficients may have to be modified by two other propagation effects which can acquire first order status. The first of these and the second first order effect required to be known, is dependence of the coefficients on past signals that have propagated along the line (frequency dependence of coefficients). The second of these, and the third first order effect required to be known is dependence of the coefficients on signals propagating in the present (non-linear propagation).

This thesis is mostly concerned with the first two of these effects and is therefore divided into two major parts. The first is concerned with the constant coefficient T.L.E.s and their application to the power system situation. This is labelled Linear Transmission Line Analysis and is subdivided into thirteen sections. The second part concerns the T.L.E.s with frequency dependent coefficients. This is labelled Transmission Line Analysis with Frequency Dependent Parameters and is subdivided into three sections. A third part of this thesis is concerned with a global assessment of progress towards achieving a complete first order transmission model, consists of one section and is labelled Reflections on Transmission Lines.



All computing carried out in relation to this thesis was done on an IBM 360/44 digital computer and programmed in Fortran IV. Graphs were plotted from computer output on a Calcomp plotter.

P A R T 1

LINEAR TRANSMISSION LINE ANALYSIS (L.T.L.A.)

## SECTION 1

INTRODUCTION TO LINEAR TRANSMISSION LINE ANALYSIS (L.T.L.A.)1.1) OBJECTS AND SCOPE OF L.T.L.A.

The Linear Transmission Line Analysis is solely concerned with the constant coefficient Transmission Line Equations (T.L.E.s). These equations form the most elementary representation of a transmission line. Even so, there is no general solution of these equations for arbitrary applied signals. This allows tremendous scope at any one time in the wide variety of solution options available for a given situation. In the power system situation, one is often faced with the choice of using a grossly simplified description of a transmission line (e.g. Lossless line) for which an exact solution is known, or using a more representative description of a transmission line (lossy line) for which an exact solution is not available and an approximate solution must be resorted to. In the L.T.L.A. the aim was initially to cover this breadth of solution option and, with respect to line and signal type, to rationalize the choice of which solution is most suited to any given situation. Having thus established the subject background, the aim was then to study in depth, application to the constant coefficient T.L.E.s of a hitherto little used solution technique known as the Method of Characteristics (M.O.C.), this being done with particular reference to the line types and signals involved in power system transient problems.

## 1.2) SECTION STRUCTURE OF L.T.L.A.

Originally it had been intended to write this analysis as a continuous dissertation, each development of the argument being logically derived from what had come immediately before. This proved to be an impossible ideal. The breadth of the subject material and the nature of the analysis as it developed indicated a number of boundaries across which chronological development could not progress. (Such a boundary occurs, for example, at the end of section 9). Section 8) does not follow from section 7) and is not in the mainstream of subject development but serves to fill in some necessary background material at that point. Fig. (1.1).) It was decided therefore to partition the analysis into sections of chronological development bounded at points where the development became largely self-contained, or at points to which reference would be made from more than one section in the future. The resulting subject development is illustrated diagrammatically in Fig. (1.1) where the ringed numbers are partitioned sections in the analysis.

## 1.3) OUTLINE OF L.T.L.A.

### 1.3.1) Brief Summary

The analysis is contained in sections 2) through 12) with section 1) introducing the analysis and section 13) summarizing and discussing results of the analysis. Of these, sections 2), 3), 4) and 8) are concerned with establishing the required subject background. Section 4) considers the many alternative options available to solve the T.L.E.s. It discusses them with respect to line and signal

type indicating areas in which different methods can be applied with advantage. Subject development begins at the end of section 4). Section 5) discusses the results of section 4) with respect to the types of solution offering most promise for application to power system transient problems. Section 6) derives the characteristic equations of a set of hyperbolic partial differential equations with two independent variables. The T.L.E.s belong to this class of equation. A matrix derivation is developed because of engineers' familiarity with matrices and for ease of computer programming. In section 7) the M.O.C. solution of hyperbolic partial differential equations is outlined and then the results of section 6) applied to obtain solutions of the T.L.E.s. In section 9) a graphical implementation of one of the solutions derived in section 7) is developed. Section 10) investigates in detail an approximate solution technique that was first proposed in section 7). Up to section 10) the analysis is primarily concerned with single circuit transmission lines. In section 11) the techniques previously developed are applied to mutually coupled multiple circuit transmission lines. The need for a further solution development is established. This is carried out in section 12) and is then applied to the multiple circuit transmission line problem in section 11).

Sections 6), 7), 9), 10), 11) and 12) are to the best knowledge of the author, original developments in the subject area. Section 4), although not containing original development, is considered by the author to be a significant contribution to the tutorial repertoire of the subject area.

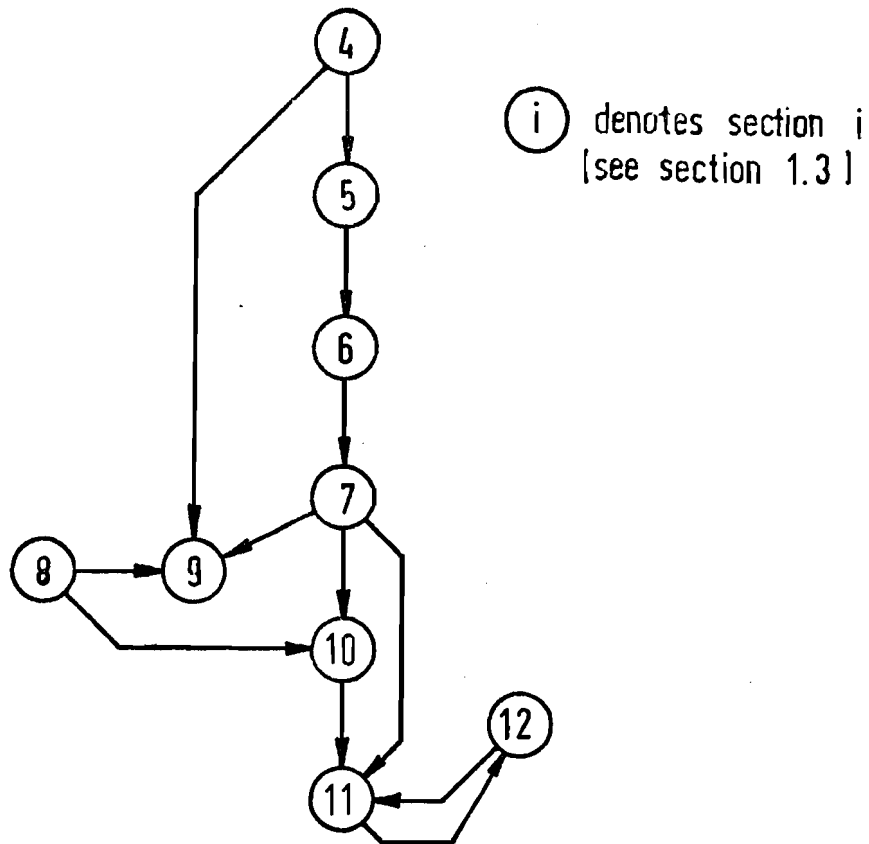


Fig. (1.1) Subject development in L.T.L.A.  
Application of M.O.C. to T.L.E.s.

### 1.3.2) Sectional Abstract of L.T.L.A.

- 1) This section forms the introduction to the L.T.L.A. The scope and objects of the analysis are outlined. The structure and subject development of the analysis are given together with a summary of details (now being read). A brief historical note on the application of the M.O.C. to the T.L.E.s is given and the conventions, abbreviations and definitions used in the analysis are listed.
- 2) The approximate nature of the constant coefficient T.L.E.s is discussed briefly and the conditions under which their use is justified are outlined.
- 3) The T.L.E.s are derived. The conditions necessary for a solution of the T.L.E.s to exist are briefly discussed.

- 4) A review of the many methods of obtaining a solution of the T.L.E.s is given. Both exact and approximate methods are considered and the situations under which different methods can best be used are discussed. Fig. (4.1) summarizes this review.
- 5) The above review is examined with respect to the solution requirements of power system travelling wave problems. It is concluded that time domain methods of solution retain the most options available for further development. It is further concluded that the subject area would benefit from a study of the M.O.C. solution technique and its application to the T.L.E.s.
- 6) The characteristic equations of a system of hyperbolic partial differential equations with two independent variables (the T.L.E.s belong to this class of equation) are derived using a matrix formulation. This treatment transforms the partial differential equations into an equivalent set of ordinary differential equations.
- 7) An outline of how the T.L.E.s are solved using the M.O.C. is given. It is shown that for the T.L.E.s the required characteristic equations can be obtained using systematic matrix methods. The application of the M.O.C. to single circuit transmission lines is examined and the conditions under which exact solutions can be obtained are derived.

- 8) The representation of a power system transmission line by a constant parameter transmission line is considered. The assumptions involved are discussed. The propagation characteristics of this representation are examined for lossless, distortionless, low loss and series resistance lines. It is concluded that only the series resistance line is valid for both short and long term transient problems although the distortionless and low loss lines can be used for short term transient problems.
- 9) A graphical implementation of the M.O.C. solution for networks containing transmission lines with attenuation but no distortion is given. This enables the graphical solution of short term transient problems to be undertaken with reasonable accuracy. The equations used are derived in section 7). Examples illustrating the method are given.
- 10) The M.O.C. is applied to the series resistance transmission line. Approximate solutions of high accuracy are obtained by assuming functional forms for the line current along the characteristic curves. The frequently used discrete lumped resistance approximation is found to be a special case of the above solution. A comparison shows the discrete lumped resistance approximation to be the most inaccurate of the solutions considered. The alternative solutions produced are identical in form to the discrete lumped resistance approximation differing only in the values of their constant coefficients.



- 11) Application of the M.O.C. to mutually coupled multiple circuit transmission lines is considered. Firstly; direct application is studied. This is shown to have a number of inherent errors, the magnitude of which can be controlled by suitably selecting the solution grid size. The method is illustrated by solving for a mutually coupled two circuit line. It is shown that an exact solution can be obtained for dispersionless transmission. Secondly; indirect application using matrix methods and component propagation is studied. CARSON'S solution is used to provide an exact result. The M.O.C. is shown to enable more flexible specification of transmission line boundary conditions than can be obtained using CARSON'S or like solutions. The need for a line point to line point formulation of the M.O.C. which can be applied to the general lossy transmission line is established. Such a formulation when applied to the solution of a two circuit line is shown to give significantly improved results compared to those obtained by direct application of the M.O.C.
- 12) The characteristic equations of a general lossy single circuit transmission line are fabricated from the transmission line's impulse responses. The resulting equations, which can be applied directly between any two points on a transmission line, have a convolution form in which the distorting and non-distorting propagation terms are separated. Solution of these equations is studied. The effects of altering the solution time increment and accuracy of the convolution weighting functions are studied. It is shown that in some cases

the amount of past information required can be reduced without significantly degrading the quality of the solution.

- 13) The main points in the subject development of the analysis are covered briefly. A list of the solution properties of the M.O.C. applied to the T.L.E.s is given. It is concluded that in the power systems context the M.O.C. has particular application in the solution of transient travelling wave problems where it has been shown to lead to new solutions that are superior to those in current use.

#### 1.4) BRIEF HISTORICAL NOTE ON USE OF THE METHOD OF CHARACTERISTICS

The method of characteristics has found application in engineering mostly as a graphical method for solving travelling wave problems on lossless transmission systems. The method was originally given by ALLIEVI (1902) and later developed by SHNYDER (1929), ANGUS (1935) and BERGERON where it was used as a graphical method of calculating transients in penstocks. BERGERON developed a particular proficiency with the graphical method and published a number of articles in the 1930's. His applications range over a wide variety of topics including the propagation of surges on electrical transmission lines. An English translation of his work was published in 1961. (BERGERON, 1961). Later developments have not significantly extended the graphical method except in section 9) of this thesis where it has been extended to

include the effects of a transmission line attenuation.\* The method was applied in a computer program to the calculation of electrical transients by FREY (1961) and has been further applied to graphical calculation of travelling wave problems by ARLETT (1966). Finally it found application in a power system transients program by DOMMEL (1969). In the literature it is usually referred to as the 'Schnyder-Bergeron' method or simply the 'graphical method'.

It has generally been assumed that the method has application only for lossless transmission systems, this being stated as late as 1967. (BRANIN, 1967). All the examples cited above have concerned lossless transmission systems. Losses, when required, were given a lumped element representation. In the above examples treatment has in all cases been limited to single circuit transmission lines. This thesis extends application of the M.O.C. to both lossy and mutually coupled multiple circuit transmission lines.

Inclusion of frequency dependent line parameters was achieved by SNELSON (1972) where use was made of numerical inversion of the fourier transform. This is dealt with in more detail in section 16) of this thesis.

#### 1.5) ABBREVIATIONS, CONVENTIONS AND DEFINITIONS

Throughout this analysis the following abbreviations are used.

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\* Accepted for publication by the Int. Jl. Elect. Enging. Educ.: "A graphical method of solution for travelling waves on transmission systems with attenuation."

T.L.E.s	Transmission Line Equations
M.O.C.	Method of Characteristics
P.D.E.s	Partial Differential Equations
L.T.L.A.	Linear Transmission Line Analysis

Except where otherwise stated the following symbols have been used according to the convention:

$\omega$	Angular frequency
$t$	time
$x$	space coordinate - usually used to denote physical position on a transmission line
$v$	transmission line voltage - $v(x,t)$
$i$	transmission line current - $i(x,t)$ positive in the direction of increasing $x$
$R$	transmission line series resistance per unit length (ohms/metre)
$L$	transmission line series inductance per unit length (henries/metre)
$G$	transmission line shunt conductance per unit length (mhos/metre)
$C$	transmission line shunt capacitance per unit length (farads/metre)
	curve - usually in the $(x,t)$ plane, often a characteristic curve
$Z_c$	transmission line surge impedance = $\sqrt{\frac{L}{C}}$ - ohms
$a$	transmission line velocity of propagation = $\frac{1}{\sqrt{LC}}$ (metres/sec)
$\Delta$	an increment of a variable e.g. $\Delta f$ or $Df$
$D$	
$\frac{\partial}{\partial x}$	partial derivative w.r.t. $x$
$\frac{d}{dx}$	total derivative w.r.t. $x$

[ ] matrix

{ } vector - line or column

### Definitions

Transmission Line: A group of conductors, usually in close physical proximity, along which electrical signals can propagate. A single circuit transmission line consists of one go and return path thus completing one circuit. A multiple circuit transmission line has more than one go and return path and can complete more than one circuit.

Boundary (Transmission Line): Any point where the distributed nature of the circuit parameters is interrupted. (Includes transmission line terminals, loading a transmission line with lumped circuit elements, junction point between two transmission lines with different characteristics etc.)

## SECTION 2

THE APPROXIMATE NATURE OF THE TRANSMISSION LINE EQUATIONS

The classical approach to transmission line analysis considers the transmission phenomena to be completely determined by the self and mutual series impedances and the self and mutual shunt admittances of the conductors. As a consequence the phenomena are completely specified in terms of the propagation constants and the corresponding characteristic impedances of the possible modes of propagation.

Although simple and of great value the above approach is not exact. To obtain an accurate solution it would be necessary to begin with Maxwell's equations and take into account such things as electromagnetic radiation, proximity effects, terminal conditions and skin effects. However, an analysis of the approximations involved (CARSON, 1928) shows that although the complete specification of a transmission system in terms of its self and mutual immittances is rigorously valid only for the case of perfect conductors embedded in a perfect dielectric, the errors introduced are of small practical significance provided that the resistivity of the conductors is much smaller than that of their surrounding dielectric and the distances between the conductors is large compared with their radii. Most systems which could be used for the efficient transmission of electrical energy would satisfy these requirements. In such cases it is possible to specify the system of conductors in terms of its self and mutual line parameters to a high degree of approximation.

The partial differential equations resulting from the above considerations will be referred to as the Transmission Line Equations (T.L.E.s). They are also known as the Telegraphers' equations. The T.L.E.s have been the object of much study in the past and continue to be so. They will be used as the starting point of, and will form the basis of, the linear transmission line analysis which follows.

## SECTION 3

DERIVATION OF THE TRANSMISSION LINE EQUATIONS (T.L.E.s)

Here the T.L.E.s will be derived according to the method outlined by ARMSTRONG (1970). In this derivation the equations of the distributed parameter network are written directly in integral form. The mean value theorem is applied to the integral and the length of line over which the integral applies is reduced to zero giving the T.L.E.s.

3.1) ANALYSIS

Consider the four circuit elements to be uniformly distributed along the transmission line and consider a finite length of that line from a point  $x = a$  to a point  $x = a + \Delta x$  where  $\Delta x > 0$ . (Fig. (3.1))

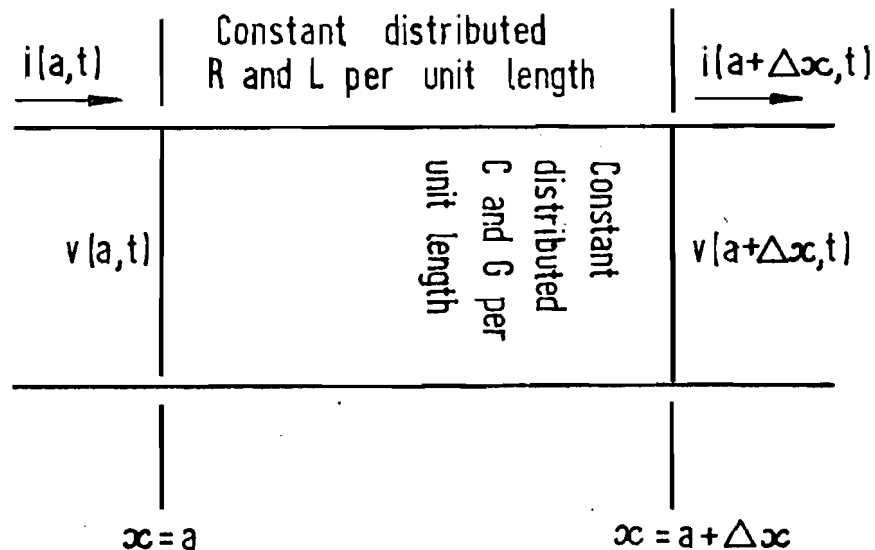


Fig. (3.1) Section of distributed parameter network representing a transmission line.



The equation of voltage balance can be written:

$$v(a,t) - v(a+\Delta x,t) = \int_a^{a+\Delta x} (R \cdot i(x,t) + L \cdot \frac{\partial i}{\partial t}(x,t)) dx \quad (3.1)$$

where  $i$  and  $\frac{\partial i}{\partial t}$  have only a finite number of discontinuities with respect to  $x$  in the range of integration. Further restricting  $i(x,t)$  and  $\frac{\partial i}{\partial t}(x,t)$  to be continuous in  $x$  over the range of integration enables the mean value theorem to be applied:

$$v(a,t) - v(a+\Delta x,t) = \Delta x (R \cdot i(h,t) + L \cdot \frac{\partial i}{\partial t}(h,t))$$

where  $a \leq h \leq a + \Delta x$

since  $\Delta x > 0$

$$\frac{v(a,t) - v(a+\Delta x,t)}{\Delta x} = R \cdot i(h,t) + L \cdot \frac{\partial i}{\partial t}(h,t)$$

Taking the limit as  $\Delta x \rightarrow 0$  gives:

$$\lim_{\Delta x \rightarrow 0} \frac{v(a,t) - v(a+\Delta x,t)}{\Delta x} = \lim_{h \rightarrow a} (R \cdot i(h,t) + L \cdot \frac{\partial i}{\partial t}(h,t))$$

and provided that the limit on the left hand side exists we obtain:

$$- \frac{\partial v}{\partial x}(x,t) = R \cdot i(x,t) + L \cdot \frac{\partial i}{\partial t}(x,t) \quad (3.2)$$

where  $x$  has replaced 'a' which was an arbitrary point.

Similarly it can be shown:

$$- \frac{\partial i}{\partial x}(x,t) = G \cdot v(x,t) + C \cdot \frac{\partial v}{\partial t}(x,t) \quad (3.3)$$

Equations (3.2) and (3.3) are the T.L.E.s of a single circuit transmission line. For multiple circuit transmission lines the variables  $i(x,t)$  and  $v(x,t)$  become vectors of variables and the coefficients become matrices of coefficients.

It will be observed that  $v$ ,  $i$ ,  $\frac{\partial v}{\partial t}$  and  $\frac{\partial i}{\partial t}$  must be continuous with respect to  $x$ . This still allows waves with sharp corners hence the necessity for considering 'provided the limit exists'. On physical grounds  $i(x,t)$  will be finite but may not be differentiable with respect to 't' everywhere. Equation (3.1) will thus be valid except for those times where  $\frac{\partial i}{\partial t}$  does not exist.

Hence it is possible that there will be a number of points on the line where the equations will not apply. Existence theorems relating to partial differential equations indicate that there exist characteristic positions where there is no solution. These positions are curves in the  $(x,t)$  plane along which discontinuities can be propagated. This is the basis of the Method of Characteristics, a technique of solving the T.L.E.s which will be dealt with in detail later in this analysis.

## SECTION 4

### SOLUTION OF THE T.L.E.s - A REVIEW

No general solution of the T.L.E.s for arbitrary wave shapes or combinations of line parameters is known. Exact solutions have been produced for only a few special cases. In other cases approximate solution procedures have been proposed. Consequently there are available many ways of obtaining a solution. Each has its own advantages and disadvantages relative to the others depending on the particular transmission system being analysed and the nature of the signals being considered. A survey of these techniques together with their advantages and disadvantages will assist in appreciating how a particular transmission problem might best be solved. It will also indicate in what direction improvements to existing techniques should proceed to further satisfy the requirements of this analysis.

This review is summarized diagrammatically in Fig. (4.1). It will be seen that solution techniques have been classified into two types. Frequency domain and time domain. Frequency domain solutions are those in which time is removed as an independent variable from both the network equations and the initial conditions while the equations are being solved. This can in fact be achieved by any transform pair, but in this analysis a transform parameter which can be interpreted as angular frequency is all that will be considered, hence 'Frequency domain'. Time domain solutions refer to those solutions during which no intermediate process removes time as an independent variable. These solutions

start with initial conditions expressed in time. Often these need not be pre-determined, only instantaneous values at discrete points in time being required. Operational methods of solution which produce exact solutions as functions of time have been classified as time domain methods. Those methods which produce operational expressions (in the frequency domain) requiring further processing by numerical inversion have been classified as frequency domain methods.

#### 4.1) CLASSIFICATIONS

Signals propagating along transmission lines are conveniently classified into two groups:

- 1) Steady state sinusoidal travelling waves.
- 2) All other signals.

The methods of solving the T.L.E.s for describing the propagation of these signals can also be classified into two groups:

- 1) Those in the frequency domain.
- 2) Those in the time domain.

Furthermore, it is useful to classify transmission lines themselves into two groups:

- 1) Lossless transmission lines.
- 2) Lossy transmission lines.

These classifications broadly delineate areas in which a given method of solution can more easily be applied than others and it is within this framework that the various methods of solution will be considered.

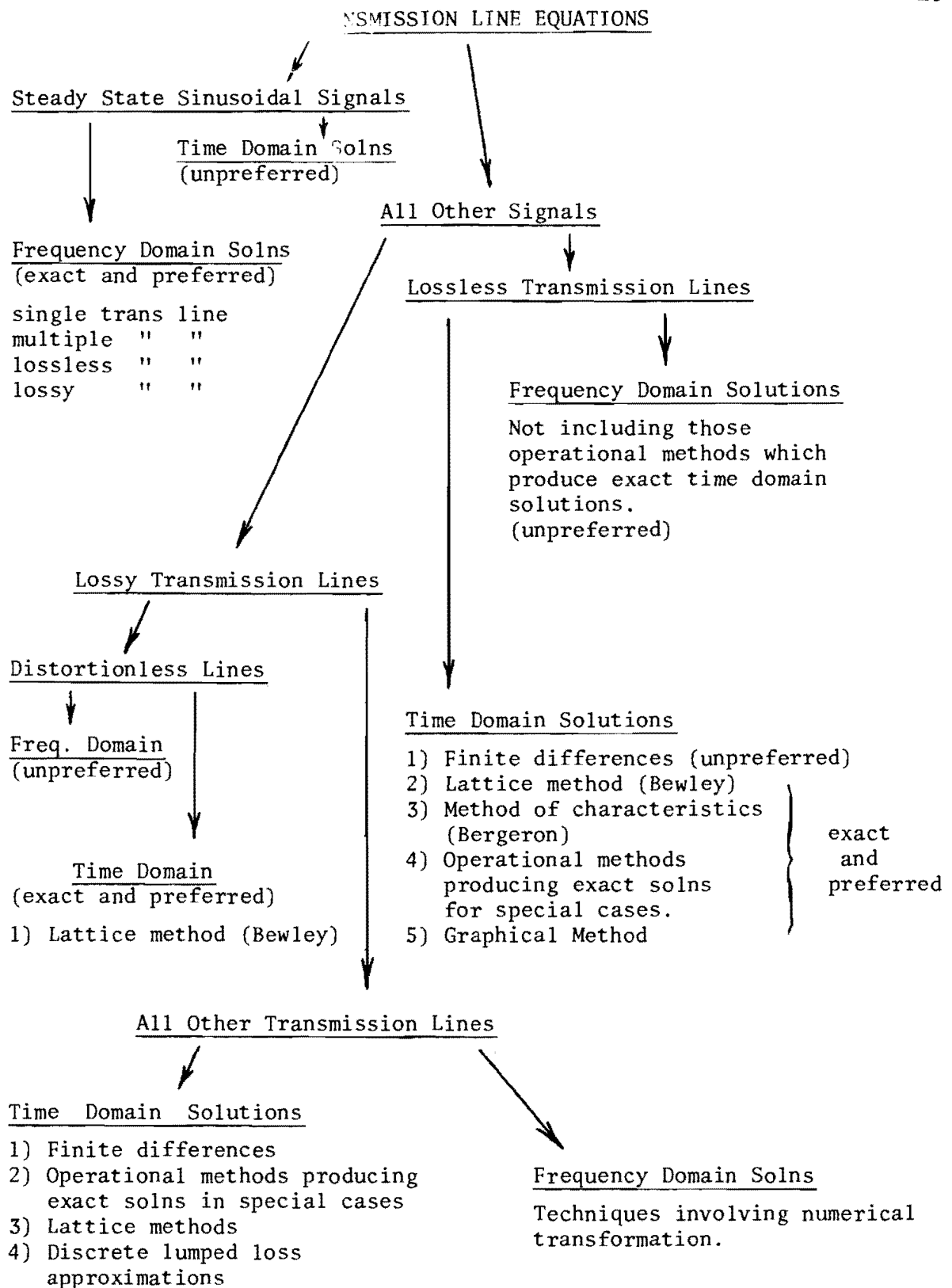


Fig. (4.1) Solution of T.L.E.s.

## 4.2) STEADY STATE SINUSOIDAL TRAVELLING WAVES

This signal is a constant frequency constant amplitude sinewave along the entire length of line. It has been in existence long enough for all transients to be absent.

### 4.2.1) Frequency Domain Solution

For a steady state sinusoidal signal the only method of solution that would normally be applied is solution in the frequency domain. An exact solution can be obtained for any linear transmission line.

#### Analysis

- (a) Single Circuit Transmission Line (ANY STANDARD TEXT e.g. MOORE, 1960)

The signal, being of exponential form, enables time variation to be removed. Frequency becomes a constant parameter and the T.L.E.s reduce to a simple ordinary differential form.

$$\frac{d^2V}{dx^2} = \gamma^2 V \quad (4.1)$$

where  $\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$  is the propagation constant.

This enables propagation to be simply specified in terms of an attenuation and phase change constant, the real and imaginary parts of the propagation constant

$$\gamma = \alpha + j\beta$$

$\alpha$  is the attenuation constant

$\beta$  is the phase change constant.

Voltage and current expressions along the line involve only exponential variation. e.g. for an infinite line  
 $V = Ae^{-\alpha x}e^{-j\beta x}$ .

(b) Multiple Circuit Transmission Lines

For these lines, instead of obtaining a single ordinary differential equation (eqn (4.1)) a set of mutually coupled differential equations is obtained.

$$\frac{d^2\{V\}}{dx^2} = [Z][Y]\{V\} = [P]\{V\} \quad (4.2)$$

where  $[Z]$  is the line series impedance matrix

$[Y]$  is the line shunt admittance matrix

$\{V\}$  is a vector of line voltages.

The classical method of solving these equations is by elimination and then integration. This is an involved and tedious process. The application of matrix methods (WEDEPOHL, 1963) simplifies the problem considerably. Here the line variables are transformed to a set of component variables according to relation (4.3):

$$\{V\} = [S]\{V_c\} \quad (4.3)$$

Substituting (4.3) into (4.2) gives:

$$\frac{d^2\{V_c\}}{dx^2} = [S]^{-1}[P][S]\{V_c\} \quad (4.4)$$

The transformation is carefully chosen so that the resulting component equations (eqns (4.4)) contain no mutual coupling (i.e. so that the coefficient matrix  $[S]^{-1}[P][S]$  is a diagonal matrix). Each component then appears to be

travelling along a single circuit transmission line and can be solved accordingly. Having thus obtained the solution for each of the component variables, the line variables (voltages in this case) can be recovered from relation (4.3). These components are often referred to as modal components. This transformation to diagonal form can always be found for equations (4.2).

An alternative approach is to further transform equations (4.2) with respect to  $x$  (BATTISSON, 1967). This gives a set of simultaneous algebraic equations to be solved.

#### 4.2.2) Time Domain Solutions

Time domain techniques can also be applied to the steady state sinusoidal signal but have little to offer when compared with the frequency domain alternative. They would admit exact solutions only for dispersionless transmission (i.e. transmission along lossless or distortionless lines) and would be considerably more difficult to implement. Multiple circuit transmission lines are more difficult to handle. Except for a few special cases the transformation to diagonal form cannot be achieved and alternative (usually more involved) techniques are required.

#### 4.3) ALL OTHER SIGNALS

The general frequency domain method of solution about to be outlined applies, with minor modifications to all subsequent sections of this review. Further subdivision of this review into various categories arises as a result of the properties of time domain solutions.



#### 4.3.1) Frequency Domain Solutions - General

Frequency domain methods for this situation become considerably more complex than for the case of a steady state sinusoidal signal. It is necessary to analyse the signal being applied into its frequency components. For linear systems this is usually easily achieved. The network equations are then solved for each component frequency using the analysis of a steady state sinusoidal signal. This gives a frequency spectrum of the solution which is then transformed back into the time domain. Except for the most simple problems, the inverse transformation will need to be done numerically. Thus a digital computer is required for this method.

Multiple circuit transmission lines can be handled as for steady state sinusoidal frequencies. Modal components are mostly used. For each frequency the transformation to modal components, the component solutions and the inverse transformation is carried out. Unfortunately the transformation to modal components is usually frequency dependent. Consequently a different transformation must be found for each component frequency. A few combinations of line parameters enable transformations to be found that are frequency invariant. In these cases the computation required is somewhat reduced. Occasionally analyses have been carried out where the transformations are assumed to be independent of frequency when in fact they are not. e.g. (BICKFORD, 1967), (McELROY, 1963). This should be treated with care as it has the effect of introducing a variation of the system parameters with frequency which would not otherwise be present. WEDEPOHL (1969) is an example of where this

technique has been correctly applied.

### Analysis

A number of integral transforms could be used to solve the T.L.E.s. The advantages of one where the transform parameter can be interpreted as angular frequency will have already become apparent. This suggests use of the complex fourier transform. Since, however, this transform does not converge for such useful functions as steps or ramps it is modified by including a real part in the exponent of the exponential chosen large enough to ensure convergence. This gives the transform pair

$$\bar{f}(a+j\omega) = \int_0^{+\infty} f(t) e^{-(a+j\omega)t} dt \quad (4.5)$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \bar{f}(a+j\omega) e^{(a+j\omega)t} d\omega \quad (4.6)$$

where  $f(t) = 0$  for  $t < 0$ .

For realistic problems convergence is assured if  $a > 0$ .

When equation (4.6) is evaluated numerically it is necessary to truncate the range of integration. This can give rise to large and sustained oscillations in the solution. It is shown (DAY, 1966 -1) that these can be removed by multiplying the integrand by a Standard  $\sigma$  factor.

$$\sigma = \frac{\text{Sin}(\pi\omega/\Omega)}{\pi\omega/\Omega} \quad (4.7)$$

where  $(-\Omega, \Omega)$  is the range of the truncated integration.

However, use of this standard  $\sigma$  factor, while removing these oscillations decreases the rate of rise of the solution and

has the effect of flattening the solution wavefronts. The standard  $\sigma$  factor can be altered to reduce this effect. The resulting  $\sigma$  factor is known as the Modified  $\sigma$  factor (DAY, 1966 - 1).

Often in evaluating equation (4.6) the path of integration will lie close to poles in the integrand. This can require very small step lengths in order to evaluate the integral. Here the factor 'a' can be given a value that removes the path of integration from these poles thus enabling a larger step length to be used. It is shown (DAY, 1966 - 2) that the integrand can be smoothed by choosing  $at \approx 1$ .

The numerical inversion is thus performed on:

$$f(t) = \frac{e^{at}}{2\pi} \int_{-\Omega}^{\Omega} \bar{F}(a+j\omega) \cdot e^{j\omega t} \cdot \frac{\sin(\pi\omega/\Omega)}{\pi\omega/\Omega} d\omega \quad (4.8)$$

where  $at \approx 1$ .

This approach was first applied to the solution of electrical transmission networks by BATTISSON (1967).

### Application

The application of this technique in conjunction with modal components to the solution of electrical transmission networks is outlined in flow diagram form in Fig. (4.2).

### Comment

Solution in the frequency domain as outlined above has the following favourable features:

- (i) The state of the system at any time can be calculated with a single direct computation over the time interval concerned.

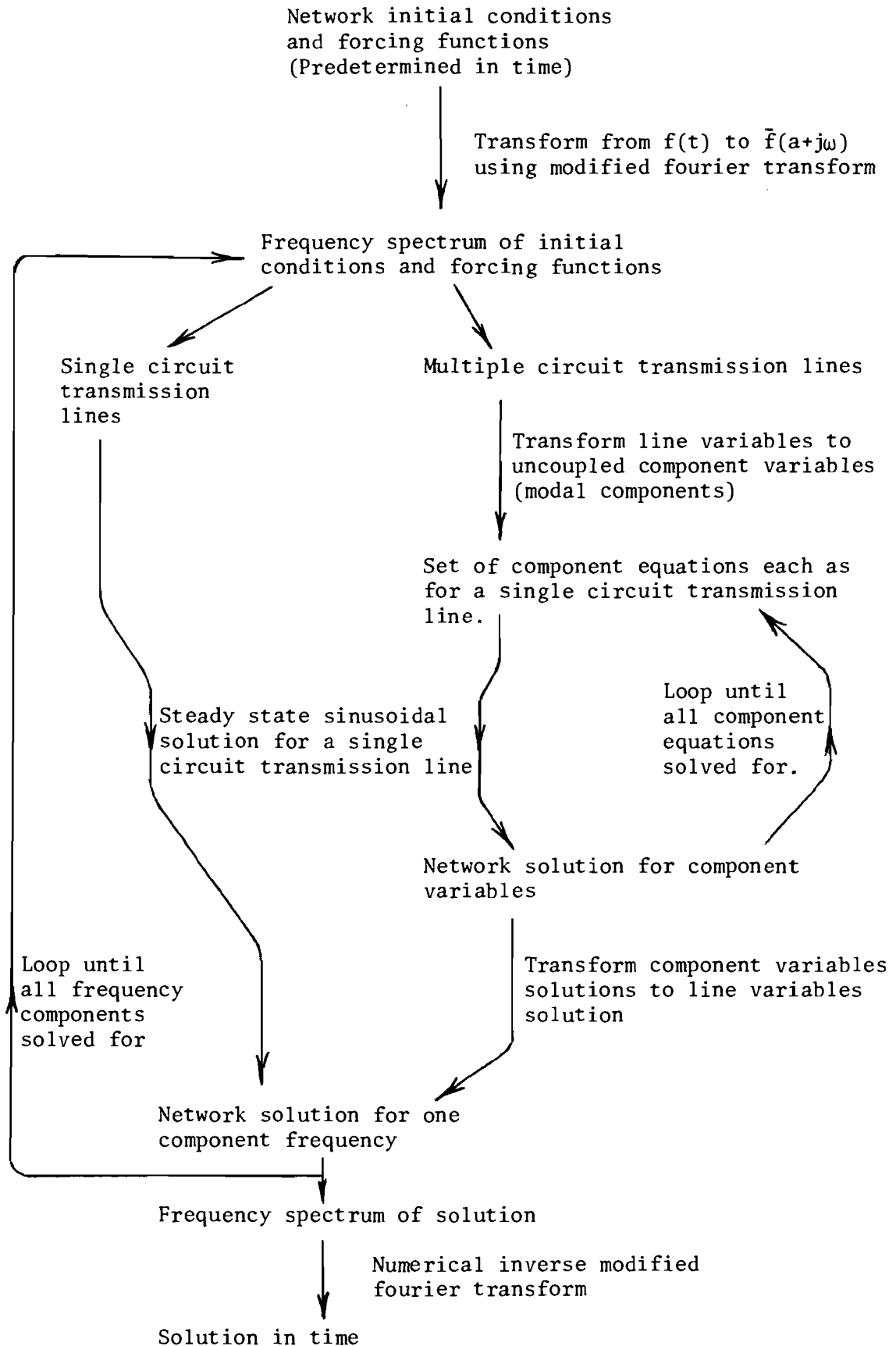


Fig. (4.2) Frequency domain solution for transmission networks using modified fourier transform.

- (ii) Only points of interest in the network need be calculated.
- (iii) Any parameter variation with frequency is simply incorporated. Thus skin and earth conduction effects are readily included in a transmission line analysis.
- (iv) The method is general and can be applied with equal felicity to any linear system regardless of losses or parameter combinations.
- (v) A solution for multiple conductor lines can always be found using this approach although the computation involved can be considerable.

The method also has a number of drawbacks:

- (vi) It cannot be used for non-linear transmission. Non-linear network elements can, however, be handled by piecewise linearization, interpolation and referral back to the time domain to find points of discontinuity (WEDEPOHL, 1970).
- (vii) The method is most suitable where the applied signals are predetermined in time and their transforms can be obtained. However, situations involving switching functions can be handled by numerically finding the transform from the problem solution where necessary (WEDEPOHL, 1970). Although Wedepohl has extended the technique to handle non-linearities which can be considered to be piecewise linear there has resulted a corresponding increase in solution complexity and the

technique is still not as inherently suitable for such problems as alternative time domain techniques.

- (viii) Although accurate the solutions are not exact. It is possible for certain conditions to obtain an exact solution using alternative techniques, some of which require considerably less computation.

#### 4.3.2) Time Domain Solutions

Time domain methods have inherent advantages for the general signal. Boundary conditions consisting of any function of time can be applied to the solution directly. Initial conditions, in general, require no special formulation and can usually be accommodated easily. A list of advantages and disadvantages typical of time domain methods is given in section 4.8) of this review. This can be compared with the frequency domain solution properties given in section 4.3.1). Such a comparison must be treated with care, however, as not all time domain methods realize these properties to the same degree and in some cases exceptions can occur.

Time domain methods will now be treated in more detail. In doing so it is useful to distinguish between lossless and lossy transmission lines and in the latter case to consider separately the special case of a distortionless line.

#### 4.4) LOSSLESS TRANSMISSION LINES

Bearing in mind the fact that lossless transmission lines do not exist, there are still many occasions where a real system may approximate very closely to a lossless system or for reasons of simplicity it is advantageous to treat it as such.

Solution of these lines by frequency domain methods requires no further comment other than that for multiple circuit transmission lines the transformation to modal components is constant, independent of frequency. Most time domain methods yield exact solutions for this case although some require more organization than others.

##### Time Domain Solutions

#### 4.4.1) Travelling Wave Solution - Classical

For a single circuit transmission line with no losses, the solution of the T.L.E.s is (BEWLEY, 1963):

$$\begin{aligned} v &= f_1\left(t + \frac{x}{a}\right) + f_2\left(t - \frac{x}{a}\right) \\ i &= -\sqrt{\frac{C}{L}} f_1\left(t + \frac{x}{a}\right) + \sqrt{\frac{C}{L}} f_2\left(t - \frac{x}{a}\right) \end{aligned} \quad (4.9)$$

where 'a' is the velocity of propagation of the line.

Function  $f_1$  represents a travelling wave propagating backwards along the line and  $f_2$  a travelling wave propagating forward along the line. For an infinite line  $f_2$  would be the signal applied to the line end and  $f_1$  would be zero.

#### 4.4.2) Bewley's Lattice Method (BEWLEY, 1963)

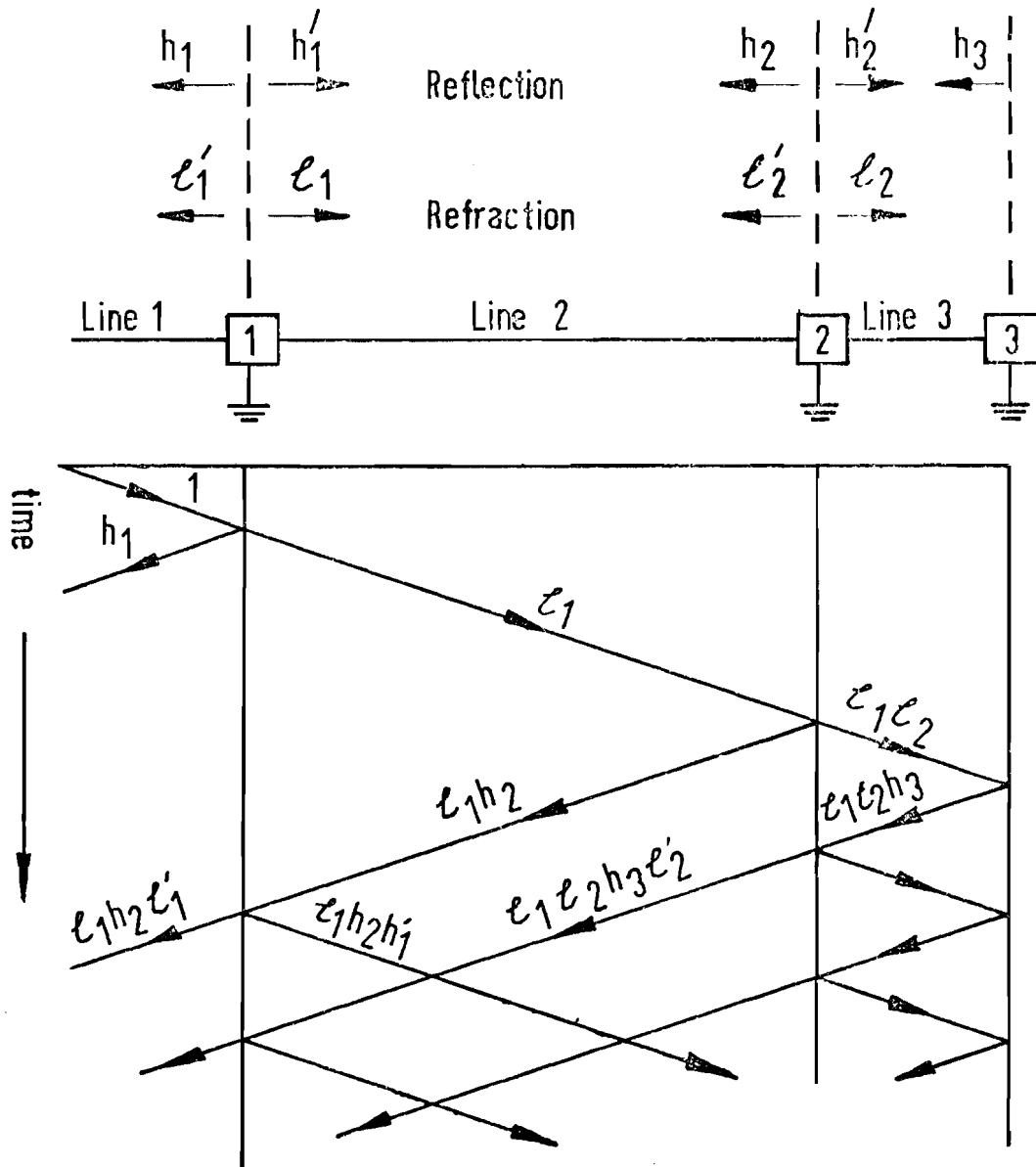
Where there are networks of interconnected transmission lines or where reflections occur, solutions of the form of equations (4.9) rapidly become cumbersome and accumulate large numbers of terms. It becomes difficult under these circumstances to trace back in time the sources of the many travelling waves that develop or to see easily at what points in time and at what positions on the lines waves will coincide additively to produce large voltages etc. To overcome these difficulties L.V. Bewley devised a semigraphical method of organizing a solution using a Lattice Diagram.

##### Construction

Consider the interconnected transmission lines shown in Fig. (4.3). To construct a lattice diagram the junctions of the transmission lines are positioned at intervals equal to the transit times of the waves on each line. A suitable vertical time scale is then chosen. (Positioning the line junctions in this manner instead of positioning according to actual line lengths has the advantage that all diagonals have the same slope regardless of the differing line properties.) The loci of the travelling wave fronts are then drawn into the diagram. In Fig. (4.3) a unit step was travelling along line 1 before it impinged on junction 1. The resulting series of waves has been drawn in. The size of each step wave is drawn above the loci of that wave.

Computation for very large and complex networks using this method can often be reduced by representing the more remote parts of the network by their step responses (BICKFORD, 1967).





$h$  is the reflection factor for waves approaching from the left

$h'$  is the reflection factor for waves approaching from the right

$l$  is the refraction factor for waves approaching from the left

$l'$  is the refraction factor for waves approaching from the right

Fig. (4.3) Example lattice diagram for a lossless single circuit transmission system.

### Comment

- (i) In such a lattice diagram the total potential at any position at any instant in time is obtained by superposing all the waves that have arrived at that position up to that instant.
- (ii) The previous history of any wave is easily traced and the position of the wave at any time is easily seen.
- (iii) In general, provided that the transmission defining functions are known and the line junction differential equations can be solved, this method will yield an exact solution.

Very good approximations to the exact solution can alternatively be obtained by representing lumped inductance and capacitance at line junctions by short transmission line stubs (BICKFORD, 1967). This eliminates line junction differential equations.

#### 4.4.3) Method of Characteristics

The T.L.E.s are classified in partial differential equation theory as being hyperbolic. As such they are amenable to treatment by the method of characteristics, a standard mathematical approach to the solution of such equations. The application of this method to a lossless transmission line is also known as the method of Bergeron and Schnyder.

### Analysis

The T.L.E.s are transformed into an equivalent set of ordinary differential equations. These ordinary differential equations apply only in certain directions (known as characteristic directions) in the  $(x,t)$  plane. For lossless transmission lines they can be integrated analytically in these directions giving an exact solution.

To obtain this solution multiply equation (3.4) by  $\sqrt{\frac{L}{C}}$  and add and subtract from equation (3.3). This gives:

$$\begin{aligned} \left( \frac{\partial}{\partial x} + \sqrt{LC} \frac{\partial}{\partial t} \right) (v + \sqrt{\frac{L}{C}} i) &= -Ri - G\sqrt{\frac{L}{C}} v \\ \left( \frac{\partial}{\partial x} - \sqrt{LC} \frac{\partial}{\partial t} \right) (v - \sqrt{\frac{L}{C}} i) &= -Ri + G\sqrt{\frac{L}{C}} v \end{aligned} \quad (4.10)$$

Consider the function  $f(x,t)$

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{dt}{dx} \frac{\partial f}{\partial t} \\ \text{or } \frac{df}{dx} &= \left( \frac{\partial}{\partial x} + \frac{dt}{dx} \frac{\partial}{\partial t} \right) f \end{aligned} \quad (4.11)$$

From (4.11) it is seen that  $\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right)$  can be regarded as a differential operator which gives the total derivative of  $f(x,t)$  with respect to  $x$  in a direction  $\frac{dt}{dx} = a$  in the  $(x,t)$  plane. Substituting (4.11) into (4.10) gives

$$\begin{aligned} \frac{d}{dx} (v + \sqrt{\frac{L}{C}} i) &= -Ri - G\sqrt{\frac{L}{C}} v \\ \text{along } \frac{dt}{dx} &= +\sqrt{LC} \end{aligned} \quad (4.12)$$

$$\begin{aligned} \frac{d}{dx} (v - \sqrt{\frac{L}{C}} i) &= -Ri + G\sqrt{\frac{L}{C}} v \\ \text{along } \frac{dt}{dx} &= -\sqrt{LC} \end{aligned} \quad (4.13)$$

We have transformed the T.L.E.s into ordinary differential equations (4.12) and (4.13). Consider integrating equations (4.12) and (4.13) for a lossless line

$$\int_{x_0}^{x_1} \frac{d(v + \sqrt{\frac{L}{C}} i)}{dx} = 0 \quad (4.14)$$

along  $C_1$

$$\int_{x_1}^{x_0} \frac{d(v - \sqrt{\frac{L}{C}} i)}{dx} = 0 \quad (4.15)$$

along  $C_2$

The integrals are line integrals in the  $(x,t)$  plane and the curves  $C_1$  and  $C_2$  are shown in Fig. (4.4). It is seen that (4.14) is describing forward propagation and (4.15) is describing backward propagation along the line.

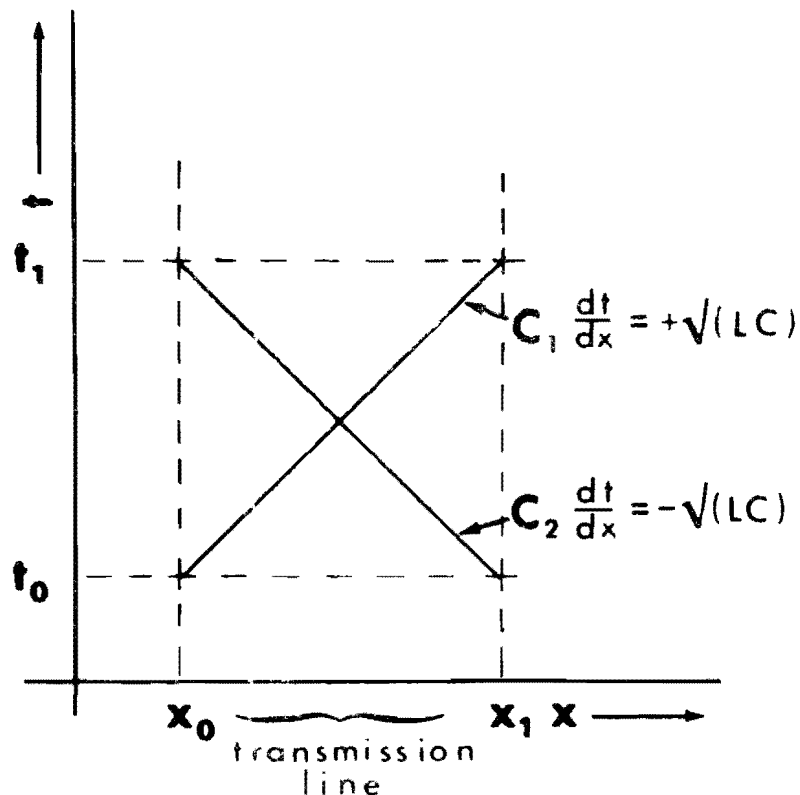


Fig. (4.4) Curves  $C_1$  and  $C_2$  along which equations (4.14) and (4.15) are integrated.

Evaluating the integrals gives

$$\begin{aligned}(v + Z_c i)_{x_1} &= (v + Z_c i)_{x_0} \quad \text{along } C_1 \\ (v - Z_c i)_{x_0} &= (v - Z_c i)_{x_1} \quad \text{along } C_2\end{aligned}\tag{4.16}$$

$$\text{where } Z_c = \sqrt{\frac{L}{C}}$$

Equations (4.16) are an exact solution of the T.L.E.s for a lossless transmission line.

#### Comment

- (i) The solution is exact.
- (ii) Equations (4.16) can be simply and effectively programmed on a computer (BRANIN, 1967) and (like Bewley's lattice method) can be used for large networks containing transmission lines.
- (iii) Reflection and refraction coefficients need not be calculated as they are automatically accounted for.
- (iv) The solution is obtained directly in terms of the values of voltage and current at one point in time thus eliminating the need to superpose the effects of the many travelling waves successively arriving and departing from the line junctions.
- (v) As the solution is defined in terms of both voltage and current at a line boundary, arbitrary terminal conditions can be handled with ease.

#### 4.4.4) Graphical Solution (ARLETT, 1966 Part 1)

The form of equations (4.16) makes them ideal for systematic programmed solution on a computer. Indeed, for large interconnected networks it becomes the only feasible means of solution. However, it is also possible to implement their solution graphically. For smaller problems this becomes a practical and useful alternative to computer solution as it requires no computing facilities and gives pictorial results directly.

#### Analysis

The key point in the graphical solution of these equations is that they are straight lines in the  $(v,i)$  plane. Consider the first of equations (4.16). This equation describes the propagation of a signal which left  $x_0$  at some time  $t_0$  and which will arrive at end  $x_1$  at a time  $t_1 = t_0 + T$  where  $T$  is the transit time of the line. Since  $(x_0, t_0)$  is a fixed point in the  $(x,t)$  plane  $(v(x_0, t_0) + Z_c i(x_0, t_0))$  is a constant and thus the voltage and current at point  $(x_1, t_1)$  are related by the equation

$$(v + Z_c i)_{(x_1, t_1)} = K \quad (4.17)$$

This is a straight line in the  $(v,i)$  plane with a slope of  $-Z_c$  passing through the point  $(v(x_0, t_0), i(x_0, t_0))$ .

Similarly it can be shown that the equation of backward propagation is a straight line in the  $(v,i)$  plane with a slope of  $+Z_c$  passing through the point  $(v(x_1, t_0), i(x_1, t_0))$ .

The graphical solution of networks containing lossless transmission lines is achieved firstly by drawing the  $(v,i)$  relations at each line terminal and then finding the

simultaneous solution of these and the equations of propagation by noting their points of intersection.

A comprehensive account of this graphical method and its application to power systems analysis is given by ARLETT (1966, Parts 1 and 2).

#### Comment

- (i) The method is exact; accuracy is limited only by drafting errors.
- (ii) The method becomes cumbersome with large systems.
- (iii) Because the solution is given by the intersection of lines in the  $(v,i)$  plane no special procedures need be adopted to obtain solutions for lines terminated with non-linear elements.

#### 4.4.5) Finite Difference Techniques

Finite difference approximations to derivatives or integrals lead to formulae from which accurate solutions are possible but from which exact solutions can never be obtained. Consequently for this case they have little to offer in comparison with the three previous solution techniques. They are however finding application in the general lossy transmission line case, particularly for multiple circuit transmission lines. They will be considered here as this is a useful point in the review to outline some of the problems associated with their use.

#### Analysis

The approaches usually used are:

- (i) Convert the T.L.E.s to a set of ordinary differential equations in time by applying discrete approximations to

the space derivatives. These are then solved using standard numerical integration techniques.

- (ii) Convert the T.L.E.s to a set of algebraic equations by applying discrete approximations in both distance and time.

Care must be exercised in the selection of the finite difference formulae. For example, it can be shown (APPENDIX (4.1)) that using the central difference approximation to the  $x$  derivatives

$$\left. \frac{\partial f}{\partial x} \right|_{x_0, t_0} = \left. \frac{f_{x_0+\Delta x} - f_{x_0-\Delta x}}{2\Delta x} \right|_{t_0} \quad (4.18)$$

and the first order forward difference approximation to the time derivatives

$$\left. \frac{\partial f}{\partial t} \right|_{x_0, t_0} = \left. \frac{f_{t_0+\Delta t} - f_{t_0}}{\Delta t} \right|_{x_0} \quad (4.19)$$

in the lossless line equations ((3.3) and (3.4) with  $R = G = 0$ ) leads to a formula that has unconditional numerical instability. If instead, approximation (4.18) had been used for both distance and time derivatives the solution would have been conditionally unstable. (The solution will be unstable if  $\left| \frac{\Delta T}{\Delta x} \right| > |\sqrt{LC}|$  (APPENDIX (4.1).))

It is also important to ensure that as the solution step increment is reduced the solution of the finite difference equation converges to the solution of the differential equations.



### Comments

- (i) Exact solutions cannot be obtained.
- (ii) The problems of numerical stability and accuracy can lead to solution step increments that are prohibitively small thus making solution a time consuming process.
- (iii) Since there exist such excellent alternative techniques, it is unlikely that a finite difference approach would ever be used for a lossless transmission line.

#### 4.4.6) Operational Methods

Where particular linear transmission networks are being considered operational methods such as the use of Laplace transforms or Heaviside operators can be used to solve the network equations. Provided that the inversions can be found exact solutions can be obtained. The method is used in section 12) and APPENDIX (8.1) of this thesis.

#### 4.4.7) Comparison of Time Domain Methods

Points of comparative interest between time domain methods for solution of electrical networks containing lossless transmission lines are discussed in table (4.1).

#### 4.4.8) Multiple Circuit Transmission Lines

For lossless transmission lines the T.L.E.s can be diagonalized directly into modal components using the same approach as outlined for steady state sinusoidal signals. The transformations required are constant. Any of the time domain techniques outlined previously can then be used to solve for the component variables.

BLM, MOC, GM are exact	FDM are not exact
Usually for BLM, MOC, GM only points of interest need be calculated.	To obtain accurate solns with FDM small soln step lengths are required. This necessitates calculation of many intermediate points not of direct interest.
BLM, MOC, GM are ideally suited to signals containing jump discontinuities. i.e. for switching functions (as often encountered in power systems analysis).	The propagation of discontinuities into the solution domain is difficult to deal with on any grid other than a grid of characteristics. Problems involving no discontinuities can be solved satisfactorily by convergent and stable FDM using rectangular solution grids.
BLM is essentially a superposition method. This requires reflection and refraction operators to be calculated and can require storage of large amounts of past information.	MOC and GM are complete solutions needing only information from one point of time in the past. Reflection and refraction operators need not be calculated as they are automatically accounted for. This simplifies solution with non-linear line terminations.
MOC and FDM would not normally be used without computers.	BLM is often used without a computer, the solution information being stored in a convenient manner on the lattice diagram. GM of course requires no computation.
GM gives pictorial solutions directly.	BLM, MOC, FDM all require calculations before pictorial information can be obtained.

TABLE (4.1) Points of comparative interest between time domain methods of solution for networks containing lossless transmission lines.

Abbreviations used:

BLM represents Bewley's Lattice Method  
MOC " " Method of Characteristics  
GM " " Graphical Method  
FDM " " Finite Difference Methods.

Alternatively finite difference methods or the method of characteristics can be applied to the T.L.E.s directly. Finite difference methods are applied in a manner identical to their application to the single circuit transmission line. In the same way as for the single circuit transmission line the method of characteristics seeks combinations of the T.L.E.s which combine the line current and voltage partial derivatives to form total derivatives in certain directions in the (x,t) plane.

#### 4.5) DISTORTIONLESS TRANSMISSION LINES

A lossy transmission line with special properties is the distortionless transmission line. This line has the unique property that the received signal is identical, except for attenuation, to the signal that was sent. To achieve this a special combination of line parameters is required. It is convenient therefore to treat this line separately as a special case rather than under the category of general lossy transmission.

For the distortionless line the parameters are restricted to that set of values that obey the relation:

$$\frac{R}{L} = \frac{G}{C} \quad (4.20)$$

This balances the series resistance voltage drop with the shunt conductance current drop maintaining a constant voltage current relation. Hence signals propagate without distorting according to:

$$v = \sqrt{\frac{L}{C}} i \quad (4.21)$$

The signal is, however, attenuated exponentially. The solution thus has the same form as for a lossless line but includes exponential attenuation.

$$\begin{aligned} v &= e^{\frac{\alpha x}{a}} f_1\left(t+\frac{x}{a}\right) + e^{-\frac{\alpha x}{a}} f_2\left(t-\frac{x}{a}\right) \\ i &= \sqrt{\frac{C}{L}} \left\{ -e^{\frac{\alpha x}{a}} f_1\left(t+\frac{x}{a}\right) + e^{-\frac{\alpha x}{a}} f_2\left(t-\frac{x}{a}\right) \right\} \end{aligned} \quad (4.22)$$

where  $\alpha = \frac{R}{L}$  and all other quantities are as defined for equations (4.9).

Equations (4.22) can be used as the defining function in Bewley's Lattice method in the same manner as for a lossless line. Now, however, an extra parameter, the attenuation of each line section, must be included in the diagram. This is easily achieved and examples for distortionless lines are given in BEWLEY (1963). As with the lossless line this gives an exact solution.

Finite difference methods have nothing new to offer when applied to this line.

#### 4.6) ALL OTHER TRANSMISSION LINES

In each of the classifications considered so far there was available at least one method of obtaining an exact solution. For the general lossy line no generally applicable exact solution is known. Using operational techniques exact expressions can be obtained for special signals applied to particular networks. These expressions usually involve

integrals of special functions such as modified Bessel functions and are difficult to evaluate. Evaluation can be achieved using tables of mathematical functions, asymptotic expansions, numerical approximations etc. (see section 4.6.1)).

As none of the techniques discussed previously will provide exact solutions for arbitrary signals no one technique now has an overriding advantage over the others. The choice of which particular solution procedure will best be used for this situation depends on such considerations as the available computing facilities, the degree of approximation that can be tolerated, the type of signal being applied, the presence of non-linearities, frequency dependent parameters, whether or not multiple circuit transmission lines are being examined etc.

A transmission line which belongs to this class and which has particular application is a line with series resistance but no shunt conductance. This line is a commonly used linear representation of an overhead power system transmission line. For this reason it is proposed to use this line to illustrate application of the solution methods that follow.

#### 4.6.1) Unit Impressed Step on an Infinite Line - Carson's Solution

J.R. CARSON (1926) used Heaviside operational calculus to obtain the response of an infinite transmission line to a unit impressed step of voltage. His solution is:

$$\begin{aligned}
\left. \begin{aligned} v(x,t) &= 0 \\ i(x,t) &= 0 \end{aligned} \right\} \quad \text{for } t < \frac{x}{a}, \quad x \geq 0 \\
v(x,t) &= e^{-\frac{\rho x}{a}} + \frac{\sigma x}{a} \int_{\frac{x}{a}}^t \frac{e^{-\rho T} I_1(\sigma \sqrt{T^2 - (x/a)^2})}{\sqrt{T^2 - (x/a)^2}} dT \\
i(x,t) &= \frac{1}{Z_c} e^{-\rho t} I_0(\sigma \sqrt{t^2 - (x/a)^2}) \\
&\quad + aG \int_{\frac{x}{a}}^t e^{-\rho T} I_0(\sigma \sqrt{T^2 - (x/a)^2}) dT \\
&\quad \text{for } t \geq \frac{x}{a}, \quad x \geq 0
\end{aligned} \tag{4.23}$$

where  $I_1$  and  $I_0$  are modified Bessel functions of the first kind and

$$\begin{aligned}
a &= \frac{1}{\sqrt{LC}} \\
\rho &= \frac{1}{2} \left( \frac{R}{L} + \frac{G}{C} \right) \\
\sigma &= \frac{1}{2} \left( \frac{R}{L} - \frac{G}{C} \right) \\
Z_c &= \sqrt{\frac{L}{C}}
\end{aligned}$$

This response when  $G = 0$  is shown in Fig. (4.5). For the purposes of illustration  $Z_c$  was chosen to be 1 p.u. The p.u. system of HEDMAN (1971) has been used (see APPENDIX (13.1)) and the voltage integral was calculated numerically. It is seen that the response differs from the more simple lossless and distortionless lines in that:

- 1) The line input current is a decreasing function of time.
- 2) The voltage and current are related by the surge impedance,  $Z_c$ , only at the toe of the wave. Thereafter the voltage and current become dissimilar.

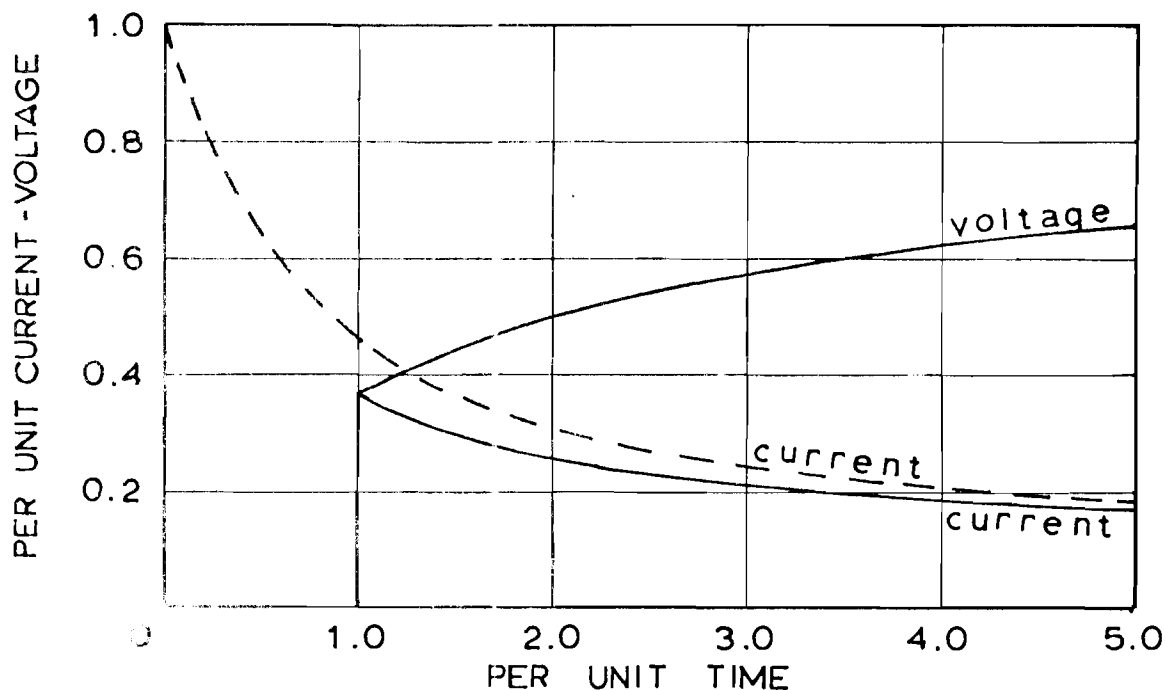


FIG (4.5) CARSON'S SOLUTION with  $G = 0$

—  $x = 1$  P.U.

---  $x = 0$  P.U.

A more detailed examination of the propagation characteristics of these equations can be found in (CARSON, 1926) and (HEDMAN, 1971) (see Section 8)). The discussion of (HEDMAN, 1971) quotes an extended form of Carson's solution where the voltage step was applied to the line through a source resistance.

The expressions (4.23) can be simplified for small or large values of time by substituting appropriate approximations to the special functions for small or large values of their arguments. For example, consider  $v(x,t)$  for small values of  $t^2 - (x/a)^2 < 1$  say. By noting that for small values of  $w$ ,  $I_1(w) \approx \frac{w}{2}$ , (ABRAMOWITZ, 1968), the integral expression can be evaluated giving

$$* \quad v(x,t) \approx e^{-\frac{\rho x}{a}} + \frac{\sigma^2 x}{2\rho a} (e^{-\frac{\rho x}{a}} - e^{-\rho t}) \quad (4.24)$$

which is valid for short periods of time after the wavefront has passed.

Considering the case for  $G = 0$  and  $t \gg \frac{x}{a}$  the asymptotic expansion of  $I_1(w) = e^w / \sqrt{2\pi w}$  (ABRAMOWITZ, 1968), can be used to evaluate the expression giving

$$* \quad v(x,t) = \operatorname{erfc} \left( \frac{x}{a} \sqrt{\frac{\rho}{2t}} \right) \quad (4.25)$$

which is valid for large times after the wavefront has passed.

Carson's solution represents a convenient standard against which the performance of more simple approximate methods of solution can be assessed.

#### 4.6.2) Bewley's Lattice Method

Bewley's lattice method can be used provided that suitable defining functions can be found. Situations involving perfect short and open circuit conditions should not be difficult to handle where, for example, Carson's solution may be suitable. For other cases however, implementation will be much more involved and suitable defining functions difficult to find.

#### 4.6.3) Finite Difference Methods

Finite difference procedures have had, in previous classifications, to compete with exact solutions. Now they no longer labour under such a disadvantage. The problems of

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\* See APPENDIX (4.2)



numerical stability and convergence are still present and should be considered in any finite difference approximation.

Often low order difference approximations are used. These are simple to apply but require small step lengths to obtain accuracy and to maintain stability. The need for small solution step lengths has traditionally weighed heavily against use of finite difference approximations. Means of increasing the permissible solution step length have therefore featured prominently in extending the application of these techniques. One example of this will be considered in detail.

#### Analysis (TALUKDAR, 1972)

Here the approach outlined in 4.4.5) (i) is used. The position derivatives are eliminated using a central difference approximation, leaving a set of ordinary differential equations in time. However, instead of integrating these equations immediately (as is traditionally done) they are further transformed into a related problem before the integration is carried out. The transformation is chosen so that the related problem is more tractable to numerical solution than the original problem. From the solution of the related problem the solution of the original problem is deduced.

TALUKDAR writes the resulting ordinary differential equations (after eliminating spatial derivatives) as the initial value problem:

$$\begin{aligned}\frac{dY(t)}{dt} &= F(Y(t), t) \\ Y(0) &= Y_0\end{aligned}\tag{4.26}$$

where  $F(Y(t), t) = Q \cdot Y(t) + \Gamma(t)$

$Y$  is a vector whose elements are unknown voltages and currents.

$\Gamma$  is a vector containing information about the line boundary conditions

$Q$  is a matrix

$Y_0$  the known initial values of  $Y$  (line initial conditions).

The equations (4.26) are transformed according to:

$$Z(t) = e^{-At} Y(t)$$

where  $A$  is a square matrix and the related initial value problem is:

$$\begin{aligned} \frac{dZ(t)}{dt} &= G(Z(t), t) \\ Z(0) &= Y_0 \end{aligned} \tag{4.27}$$

where  $G(Z(t), t) = e^{-At} [F(e^{At} Z(t), t) - Ae^{At} Z(t)]$ .

Choosing  $A$  so that

$$A = \left. \frac{\partial F(Y, t)}{\partial Y} \right|_{\substack{Y=Y_0 \\ t=0}} = Q \Big|_{\substack{Y=Y_0 \\ t=0}}$$

makes the related initial value problem (4.27) more amenable to numerical solution than problem (4.26). The solution of the original problem is recovered using the inverse transformation

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$$Y(t) = e^{At} Z(t)$$

Effectively this transformation smooths  $Y(t)$  to give  $Z(t)$ . This enables  $Z(t)$  to be accurately estimated using much larger step sizes than  $Y(t)$  would allow. Equations (4.27) can be solved by any standard numerical integration method. TALUKDAR uses a Runge Kutta method. To be successful this method requires the use of approximations to the exponential matrix that are stable. An algorithm for the generation of such an approximation is given by TALUKDAR (1972).

Another approach, used to obtain a larger solution step size is to use a higher order difference approximation. This can however introduce new difficulties such as requiring to know the derivatives of the line variables at the line terminals etc. An example of this approach is given by RAU (1972).

#### Comment

- (i) The solution is not exact but can be made very accurate.
- (ii) As the line variables are directly available non-linear propagation can be handled.
- (iii) The method can be applied directly to multiple circuit transmission lines. This provides a means (the only means in this review) of solving for non-linear propagation along multiple circuit transmission lines.
- (iv) Usually calculation must be carried out at many intermediate points on the transmission line.

#### 4.6.4) Discrete Lumped Loss Approximation

A frequently used method of obtaining an approximate solution for the general lossy transmission line is to approximate the continuous reflections and refractions that occur on the actual line by discrete reflections and refractions on a lumped loaded lossless line. Here a cascaded series of lossless transmission line segments separated by lumped loss elements is formed.

##### Analysis

This approach can be formulated in two ways. Using the Bewley lattice approach the solution can be obtained in terms of voltage only or current only. HEDMAN (1971) has applied this technique to both single and multiple circuit series resistance transmission lines. Using the method of characteristics the solution is found in terms of both voltage and current simultaneously. The method of characteristics approach is treated in detail in section 10.1) of this analysis. The alternative superposition lattice approach will be considered here.

As an example of solution using this method consider an approximation to a single circuit series resistance transmission line. The approximation is illustrated in Fig. (4.6).

For an incident wave arriving at the junction of a lossless line segment and a lumped resistance the coefficient of reflection is

$$\Gamma = \frac{RAZ_0}{ZZ_0 + RAZ_0} \quad (4.28)$$

and the coefficient of refraction is

Transmission line with continuously distributed resistance  $R$  per unit length.

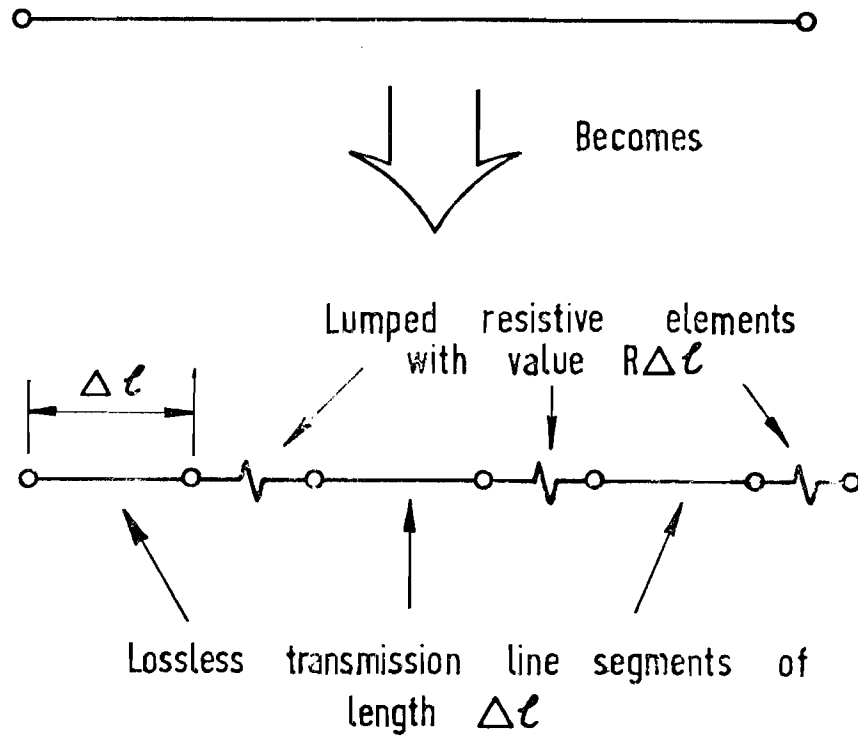


Fig. (4.6) Discrete resistance approximation of a transmission line with distributed series resistance.

$$t = \frac{2Z_c}{2Z_c + R\Delta \ell} = \left(1 + \frac{R\Delta \ell}{2Z_c}\right)^{-1} \quad (4.29)$$

where  $Z_c$  is the surge impedance of the transmission line. The reflection and refraction lattice is set up over the length of transmission line (as in section 4.4.2)) and the total solution obtained by superposition. At the  $n^{\text{th}}$  section the toe of a forward travelling wave will have the value

$$V_n = \left(1 + \frac{R\Delta \ell}{2Z_c}\right)^{-n} V_0 \quad (4.30)$$

Multiple circuit transmission lines can be handled by forming a discrete resistance network instead of having a single resistance (HEDMAN, 1971). However, the method, as outlined by HEDMAN, does not take into account the differing velocities of the various modes of propagation.

Comment

- (i) The solution is not exact. The toe of the wavefront is always high and the surge impedance of the approximation is different from that of the actual line (see section 10.2.2)(d) of this analysis).
- (ii) As with other time domain techniques, this approach requires calculation to be carried out at intermediate points on the line.
- (iii) The method can be applied to multiple circuit transmission lines.

4.6.5) Frequency Domain Methods

Frequency domain methods now become viable alternatives to time domain methods for suitable signals. They have the distinction of always admitting a solution for multiple circuit transmission lines regardless of the line's physical geometry. The computation required need not be large in comparison with time domain alternatives as these now require either evaluation of complex expressions or calculation at intermediate points which are not of direct interest.

However, the presence of non-linearities can still thwart this method and alternative time domain methods must be used.

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#### 4.7) MULTIPLE CIRCUIT TRANSMISSION LINES - GENERAL

##### 4.7.1) Physical Description of Propagation

A signal applied to such a line will propagate as a number of distinct waves which sum to produce the applied signal at its point of application. In general, these waves will travel at different velocities, have different rates of attenuation and different surge impedances. The signal at any point on the transmission line is the sum of these waves at that point. Owing to separation of the waves and their different propagation characteristics it is apparent that the signal arriving at the far end of the line need bear little resemblance to the original applied signal (BEWLEY, 1963).

##### 4.7.2) Modal Components

This is one of the most frequently used methods of handling multiple circuit lines. It is particularly suitable as it not only simplifies the problem of solution but also parallels the physical properties of propagation by separating the applied signal into a number of component signals. The method achieves its simplification by finding component signals that are uncoupled (4.2.1) part (b)). In general such components can be found only at one frequency. Thus frequency domain methods can always utilize this approach. There do exist however, a number of transmission lines on which component waves remain uncoupled at all frequencies. For these transmission lines the method of modal components can be utilized in conjunction with time domain techniques.

### Special Cases of Frequency Independent Transformations

Here it is possible to find a single transformation that will diagonalize all of the coefficient matrices in the T.L.E.s.

#### (a) Lossless Transmission Lines

The most obvious example in which a single transformation will diagonalize all the coefficient matrices of the T.L.E.s occurs when there is only one coefficient matrix to diagonalize. This occurs where there are no loss coefficient matrices.

#### (b) Fully Symmetric Transmission Lines

For these lines all the coefficient matrices have the same form

$$m_{ii} = m_{11}$$

$$m_{ij} = m_{12} \quad i \neq j$$

where  $m_{ij}$  are the matrix elements. The simplest example of this is a horizontal two conductor overhead line (important because of its application in h.v.d.c. power transmission systems). The three phase example corresponds to a fully transposed transmission line. It is shown by WEDEPOHL (1963) that any number of component systems can be found that will transform the equations of this line into diagonal form.

#### (c) Transient Series Resistance Line

By neglecting the internal self inductance of the lines, a series resistance line having equal resistance in each transmission circuit can be diagonalized by a single transformation. For this line the inductance and capacitance



matrices are related by:

$$K[L] = [C]^{-1}$$

where  $K$  is a scalar constant and the T.L.E.s can be written:

$$\begin{aligned} \frac{\partial}{\partial x}\{v\} &= -[L]\frac{\partial}{\partial t}\{i\} - R[I]\{i\} \\ [C]^{-1}\frac{\partial}{\partial x}\{i\} &= -\frac{\partial}{\partial t}\{v\} \end{aligned} \quad (4.31)$$

where  $[I]$  is the unit matrix and  $R$  the resistance per unit length of each transmission circuit. It is seen that equations (4.31) can be transformed into diagonal form by setting

$$\{v\} = [S]\{v_c\}$$

$$\{i\} = [S]\{i_c\}$$

and choosing  $[S]$  such that

$$[S]^{-1}[L][S]$$

is a diagonal matrix.

#### Comment

- (i) The method of modal components can always be applied for steady state sinusoidal signals as in the frequency domain the T.L.E.s have only one coefficient matrix.
- (ii) The method can only be applied in the time domain where component signals can be found that are independent of frequency. Three such cases have been listed. Examples (a) and (b) are frequently used in the literature. Example (c) has special significance

in that it is a linear representation of an overhead power system transmission line under transient conditions.

- (iii) Since the solution is obtained in terms of components the method cannot handle non-linear transmission.
- (iv) It is possible to carry out a form of selective filtering by modifying the component waves differently. This has particular application in transmission lines with ground returns.

#### 4.7.3) Direct Application of Time Domain Techniques

This approach is necessary if non-linear transmission is being considered or if a time domain solution is desired on a transmission line which does not admit a frequency independent transformation to modal components. To this end finite difference techniques have recently been attracting attention (TALUKDAR, 1972), (RAU, 1972). Direct application of finite difference techniques to multiple circuit lines serves only to increase the number of equations that must be solved. The level of approximation remains unaltered.

#### 4.8) FEATURES TYPICAL OF TIME DOMAIN SOLUTION PROCEDURES

Time domain methods of solution have now been considered in some detail and a number of characteristic features will have become apparent. The following properties can be listed as favourable features.

- (i) Signals consisting of any function of time can be applied to the solution directly.
  - (ii) Initial conditions, in general, require no special formulation and can be accommodated easily.
-

- (iii) The values of the solution variables are available directly making solution of non-linear problems possible.
- (iv) Exact solutions can always be obtained for lossless and distortionless transmission lines. Exact solutions are not otherwise generally obtainable although accurate approximations are possible.
- (v) Often time domain methods are more simple than their frequency domain counterparts.

The following unfavourable features can also be listed.

- (vi) The effects of frequency dependent parameters are difficult to include.
- (vii) Often the use of many intermediate solution points not of direct interest is required.
- (viii) The method of modal components in multiple circuit transmission lines has limited application. The alternative, direct application of time domain techniques, is usually more involved.

#### 4.9) IN CONCLUSION

It will by now be apparent that many ways of obtaining solutions to the T.L.E.s have been devised. This has occurred because no general solution of the T.L.E.s for arbitrary waveshapes and line parameters is available. Exact solutions have been obtained for the specific cases of:

- 1) Steady state sinusoidal signals
- 2) Lossless lines
- 3) Distortionless lines
- 4) A step of voltage applied to an infinite line.

Many problems of interest involve transmission lines and signals that do not fall within the above categories and for these, approximate methods of solution must be resorted to.

## APPENDIX (4.1)

NUMERICAL STABILITY OF FINITE DIFFERENCE APPROXIMATIONS  
TO THE T.L.E.s

The stability of these approximations will be examined using the fourier series method (SMITH, 1965). This method, developed by Neumann, expresses an initial line of errors in terms of a finite fourier series and considers the growth of a function that reduces to this series for  $t = 0$  using a separation of variables solution. The errors at the solution points along  $t = 0$  are denoted by  $E(p\Delta x) = E_p$  where  $p = 0, \dots, N$  for  $(N+1)$  solution points.

$$E_p = \sum_{n=0}^N A_n e^{j\beta_n p \Delta x} \quad p = 0, 1, \dots, N \quad j = \sqrt{-1}$$

Since the finite difference equations are linear, the propagation of the error due to a single term such as  $e^{j\beta p \Delta x}$  is all that need be considered. For this the coefficient  $A$  is a constant. The function

$$E_{p,q} = A e^{j\beta p \Delta x} e^{\alpha t}$$

reduces to  $E_p$  for  $t = 0$ . Let  $t = q\Delta t$ .

$$E_{p,q} = A e^{j\beta p \Delta x} e^{\alpha q \Delta t} = A e^{j\beta p \Delta x} \xi^q$$

Provided that  $|\xi| \leq 1$  the error will not increase with  $t$ .

This function will now be examined for two finite difference approximations to the T.L.E.s.

The lossless T.L.E.s are:

$$\frac{\partial v}{\partial x} + L \frac{\partial i}{\partial t} = 0$$

$$\frac{\partial i}{\partial x} + C \frac{\partial v}{\partial t} = 0$$

Approximation (1)

(a) second order central difference approximation to  $\frac{\partial}{\partial x}$

(b) first order forward difference approximation to  $\frac{\partial}{\partial t}$

gives the difference equations:

$$\frac{v_{p+1,q} - v_{p-1,q}}{2\Delta x} + L \frac{i_{p,q+1} - i_{pq}}{\Delta t} = 0$$

$$\frac{i_{p+1,q} - i_{p-1,q}}{2\Delta x} + C \frac{v_{p,q+1} - v_{pq}}{\Delta t} = 0$$

As the error  $E_{p,q}$  satisfies the same difference equations as  $v$  and  $i$  it can be substituted directly. Let the initial error in  $v$  be  $Ae^{j\beta x}$  and  $i$  be  $Be^{j\beta x}$ . Substituting for  $v$  and  $i$  gives:

$$\frac{\rho}{2} A(e^{j\beta\Delta x} - e^{-j\beta\Delta x}) + LB(\xi-1) = 0$$

$$\frac{\rho}{2} B(e^{j\beta\Delta x} - e^{-j\beta\Delta x}) + CA(\xi-1) = 0$$

where  $\rho = \frac{\Delta T}{\Delta x}$ . Eliminating  $\frac{A}{B}$  and solving for  $\xi$  gives:

$$|\xi| = \sqrt{1 + \frac{\rho^2}{LC} \sin^2 \beta\Delta x} > 1 \quad \text{for } \rho \neq 0$$

i.e. Approximation (1) is always unstable

Approximation (2)

(a) second order central difference approximation to  $\frac{\partial}{\partial x}$

(b) second order central difference approximation to  $\frac{\partial}{\partial t}$

gives:

$$\frac{v_{p+1,q} - v_{p-1,q}}{2\Delta x} + L \frac{i_{p,q+1} - i_{p,q-1}}{2\Delta t} = 0$$

$$\frac{i_{p+1,q} - i_{p-1,q}}{2\Delta x} + C \frac{v_{p,q+1} - v_{p,q-1}}{2\Delta t} = 0$$

As before:

$$\rho A (e^{j\beta\Delta x} - e^{-j\beta\Delta x}) + LB(\xi - \frac{1}{\xi}) = 0$$

$$\rho B (e^{j\beta\Delta x} - e^{-j\beta\Delta x}) + CA(\xi - \frac{1}{\xi}) = 0$$

Eliminating  $\frac{A}{B}$  gives:

$$\xi = \pm j \frac{\rho}{\sqrt{LC}} \sin \beta\Delta x \pm \sqrt{-\frac{\rho^2}{LC} \sin^2 \beta\Delta x + 1}$$

and

$$|\xi| = \left( \frac{\rho^2}{LC} \sin^2 \beta\Delta x - \frac{\rho^2}{LC} \sin^2 \beta\Delta x + 1 \right)^{\frac{1}{2}} = 1$$

if and only if  $\left| \frac{\rho^2}{LC} \right| \leq 1$  or  $|\rho| \leq \sqrt{LC}$

i.e. Approximation (2) will be numerically stable provided that the solution grid is chosen so that

$$\left| \frac{\Delta T}{\Delta x} \right| \leq \sqrt{LC}.$$

## APPENDIX (4.2)

APPROXIMATIONS TO CARSON'S SOLUTION FOR SHORT AND LONG  
PERIODS OF TIME AFTER THE WAVEFRONT HAS PASSED

Carson's solution for  $v(x,t)$  is:

$$v(x,t) = e^{-\frac{\rho x}{a}} + \frac{\sigma x}{a} \int_{\frac{x}{a}}^t \frac{e^{-\rho T} I_1(\sigma w)}{w} dT$$

where  $w = \sqrt{T^2 - (\frac{x}{a})^2}$

when  $\sigma w$  is small, i.e.  $\sigma w < 1$ ,  $I_1(\sigma w) \approx \frac{\sigma w}{2}$  (ABRAMOWITZ, 1968)

Substituting into Carson's solution and integrating gives:

$$v(x,t) \approx e^{-\frac{\rho x}{a}} + \frac{\sigma^2 x}{2\rho a} (e^{-\frac{\rho x}{a}} - e^{-\rho t})$$

When  $G = 0$ ,  $\sigma = \rho$  and the expression reduces to:

$$v(x,t) \approx e^{-\frac{\rho x}{a}} + \frac{\rho x}{2a} (e^{-\frac{\rho x}{a}} - e^{-\rho t})$$

Putting  $G = 0$  into Carson's solution gives:

$$v(x,t) = e^{-\frac{\rho x}{a}} + \frac{\rho x}{a} \int_{\frac{x}{a}}^t \frac{e^{-\rho T} I_1(\rho w)}{w} dT$$

Using the condition that  $v(x,\infty) = 1$  gives:

$$v(x,t) = 1 - \frac{\rho x}{a} \int_t^\infty \frac{e^{-\rho T} I_1(\rho w)}{w} dT$$



Using the asymptotic expansion of  $I_1(Z)$  for large  $Z$

$I_1(Z) \approx e^Z / \sqrt{2\pi Z}$ , (ABRAMOWITZ, 1968) gives:

$$v(x,t) = 1 - \frac{\rho x}{a} \int_t^\infty \frac{e^{-\rho T} e^{\rho T (1 - \frac{1}{2} (\frac{x}{aT})^2 + \dots)}}{\sqrt{T^2 - (\frac{x}{a})^2} \cdot \sqrt{2\pi\rho} \sqrt{T^2 - (\frac{x}{a})^2}} dT$$

For large values of  $T$  this can be reduced to:

$$v(x,t) = 1 - \frac{\rho x}{a} \frac{1}{\sqrt{2\pi\rho}} \int_t^\infty T^{-\frac{3}{2}} e^{-\frac{1}{2} \frac{\rho}{T} (\frac{x}{a})^2} dT$$

Changing the variable of integration to  $Z = (\frac{x}{a}) \sqrt{\frac{\rho}{2}} T^{-\frac{1}{2}}$  gives:

$$v(x,t) = 1 + \frac{2}{\sqrt{\pi}} \int_{Z(t)}^0 e^{-Z^2} dZ$$

$$\underline{v(x,t) = \operatorname{erfc}(Z) = \operatorname{erfc} \left( \frac{x}{a} \sqrt{\frac{\rho}{2t}} \right)}$$

which is valid for  $G = 0$  and  $t \gg (\frac{x}{a})^2$ .

## SECTION 5

FURTHER DEVELOPMENT

One feature revealed by section 4) is the abundance of simple and exact techniques available for solution of the lossless transmission problem. Another feature revealed by section 4) is the lack of any simple technique to handle the general lossy transmission problem. For this problem, solution in the frequency domain requires numerical inverse transformation and sometimes numerical forward transformation. Solution in the time domain requires calculation at many points. Both frequency and time domain solutions can require the storage of large amounts of information and neither yield an exact solution. Thus it has been common to use a lossless transmission line representation to reduce to workable proportions the complexity of problems involving travelling waves in power systems. Since power system transmission lines belong to the general lossy category this represents a fairly gross approximation and as power system voltage levels have increased a need for more accurate representation has become apparent.

In seeking to develop such a representation a choice of whether to use a frequency domain or time domain approach must be made. Although both techniques are theoretically able to handle a large number of typical power system switching and transient problems, the balance of a decision must weigh in favour of a time domain approach. Typically a switching transient and surge propagation problem in a power system involves non-linear circuit elements, a changing

system topology and discontinuous signals. These are inherently more easily handled by time domain formulations. Frequency domain methods are more suited to the solution of linear problems. Problems where non-linearities can be considered to be composed of discontinuous linear segments can be solved using frequency domain methods but at the expense of increased computation. In choosing to develop a time domain formulation the difficulty of including frequency dependent parameters has been increased but the possibility of treating non-linear transmission (corona) has been retained.

Time domain methods of solution fall into two categories. In the first, voltage and current increments are propagated throughout the system. These are the superposition-based techniques and are usually organized by a lattice diagram (Bewley's Lattice Method). In the second, actual system voltages and currents are used directly (Finite Difference Techniques etc.). Although both categories can handle non-linear elements it is evident that this will be more easily achieved by the second than by the first. Furthermore the second approach allows the possibility of solving for non-linear transmission. Consequently a time domain solution of the T.L.E.s using the second approach maintains the maximum number of options available for further development. This approach will therefore be investigated in an attempt to find a simple and accurate representation of a lossy power system transmission line.

From section 4) it will be seen that only Finite Difference Techniques, section 4.6.3), and the Discrete Loss Approximation, section 4.6.4), fall into the above second

category. Of these, only the Discrete Loss Approximation has been written into a comprehensive power system electromagnetic transients program (DOMMEL, 1969). The formulation used, however, is not capable of handling non-linear transmission. Thus of the techniques currently available, only finite difference techniques maintain all the options outlined. Unfortunately Finite Difference Techniques are prone to most of the undesirable features associated with time domain solutions plus a number of undesirable features unique to themselves.

An alternative technique of the required type, available for lossless transmission but not for lossy transmission, is the Method of Characteristics, section 4.4.3). For the lossless case this technique forms the basis of one of the most flexible approaches to obtaining a solution. Its application to the lossy problem, however, appears to have been neglected. Certainly, it is commonly held that the method yields an exact solution only for a lossless transmission line (e.g. BRANIN, 1967). In the lossy situation no general exact solution is available but its application may lead to approximate solutions of a more desirable nature than those already available. It is proposed therefore to carry out a comprehensive examination of the Method of Characteristics applied to the general lossy T.L.E.s. This examination will attempt to assess the abilities and limitations of the Method of Characteristics and in particular to produce the basis of a simple representation of a lossy power system transmission line.

## SECTION 6

CHARACTERISTIC EQUATIONS OF A SYSTEM OF HYPERBOLIC PARTIAL  
DIFFERENTIAL EQUATIONS WITH TWO INDEPENDENT VARIABLES

The notion of characteristics is used in Partial Differential Equation (P.D.E.) theory to classify the different types of equation and is a key point in the understanding of that theory. The T.L.E.s belong to that subset of P.D.E.s that are classified as hyperbolic and are composed of functions of two independent variables. The method of characteristics is a standard mathematical treatment of those P.D.E.s that are hyperbolic and quasi-linear. It can thus be applied directly to the T.L.E.s. In formulating the characteristic equations of these P.D.E.s matrix notation will be used. This notation has the advantage of being easy to follow and of allowing standard matrix subroutines to be used to program the formulation on a computer.

### 6.1) TOTAL DERIVATIVES

A linear combination  $a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial t}$  of the two partial derivatives of a function  $f(x,t)$  means derivation of  $f$  in a direction given by  $\frac{dt}{dx} = \frac{b}{a}$  in the  $(x,t)$  plane. This point is central to the reasoning behind the method of characteristics and is shown in equations (4.11) of section 4.4.3).

## 6.2) MATRIX DERIVATION OF THE CHARACTERISTIC EQUATIONS

The method of characteristics seeks linear combinations of the system partial derivatives which combine to form total derivatives in the same direction in the  $(x,t)$  plane. In the process the P.D.E.s are transformed into an equivalent set of ordinary differential equations. These ordinary differential equations are known as the characteristic equations of the system.

The T.L.E.s describing wave propagation along a multiple circuit transmission line are:

$$\left. \begin{aligned} \frac{\partial}{\partial x}\{v\} + [L]\frac{\partial}{\partial t}\{i\} + [R]\{i\} &= 0 \\ \frac{\partial}{\partial x}\{i\} + [C]\frac{\partial}{\partial t}\{v\} + [G]\{v\} &= 0 \end{aligned} \right\} \quad (6.1)$$

These can be combined into the more general form:

$$[a]\frac{\partial}{\partial x}\{f\} + [b]\frac{\partial}{\partial t}\{f\} + [h]\{f\} = 0 \quad (6.2)$$

where  $\{f\} = \begin{Bmatrix} \{v\} \\ \{i\} \end{Bmatrix}$  column vector of variables  
and  $[a]$ ,  $[b]$  and  $[h]\{f\}$  are known functions of  $(x,t,\{f\})$

The  $i^{\text{th}}$  equation of (6.2) can be written:

$$L_i = \sum_{j=1}^n a_{ij} \frac{\partial f_j}{\partial x} + \sum_{j=1}^n b_{ij} \frac{\partial f_j}{\partial t} + \sum_{j=1}^n h_{ij} f_j \quad (6.3)$$

To obtain the characteristic equations of (6.2) we seek linear combinations of (6.3)

$$L = \sum_{i=1}^n \lambda_i L_i \quad (6.4)$$

such that in the differential expression  $L$  the partial derivatives of  $\{f\}$  combine to form total derivatives, each in the same direction in the  $(x,t)$  plane.

In matrix form equation (6.4) can be written:

$$\{\lambda\}[a]\frac{\partial}{\partial x}\{f\} + \{\lambda\}[b]\frac{\partial}{\partial t}\{f\} + \{\lambda\}[h]\{f\} = 0 \quad (6.5)$$

where  $\{\lambda\}$  is the row vector  $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ .

From section 6.1) we can write  $\frac{d}{dx}\{f\} = (\frac{\partial}{\partial x} + \frac{dt}{dx} \frac{\partial}{\partial t})\{f\}$ .

Substituting this into equation (6.5) gives:

$$\{\lambda\}[a]\frac{d}{dx}\{f\} + \{\lambda\}[h]\{f\} + \{\lambda\}[[b] - \frac{dt}{dx}[a]]\frac{\partial}{\partial t}\{f\} = 0$$

which can be written:

$$\{\lambda\}[a]\frac{d}{dx}\{f\} + \{\lambda\}[h]\{f\} = 0 \quad (6.6)$$

provided that the vector  $\{\lambda\}$  is chosen such that:

$$\{\lambda\}[[b] - \frac{dt}{dx}[a]]\frac{\partial}{\partial t}\{f\} = 0$$

This will be true in general only if each coefficient of  $\frac{\partial}{\partial t}\{f\}$  is zero. i.e. provided that:

$$[[b] - \frac{dt}{dx}[a]]^T \{\lambda\}^T = 0 \quad (6.7)$$

(superscript  $T$  means "the transpose of").

This is a set of  $n$  simultaneous algebraic equations in the  $n$  unknowns  $\{\lambda\}$  and a necessary and sufficient condition that non-trivial values of  $\{\lambda\}$  can be found to satisfy these equations is that the determinant of the matrix of coefficients must vanish.

$$|[b] - \frac{dt}{dx}[a]| = 0 \quad (6.8)$$

Equation (6.8) is an  $n^{\text{th}}$  degree polynomial in the quotient  $\frac{dt}{dx}$ . For a hyperbolic set of equations there will be  $n$  real and distinct roots of this polynomial.\* Let these roots be:

$$(\xi_1, \xi_2, \dots, \xi_n)$$

Equation (6.6) is valid whenever  $\frac{dt}{dx} = \xi_i$  with the corresponding  $\{\lambda\}_i$  being obtained from equations (6.7). Thus there will be  $n$  sets of two ordinary differential equations:

$$\left. \begin{aligned} \frac{dt}{dx} &= \xi_i \\ \{\lambda\}_i [a] \frac{d}{dx} \{f\} + \{\lambda\}_i [h] \{f\} &= 0 \end{aligned} \right\} \quad (6.9)$$

Equations (6.9) are known as the characteristic equations of equations (6.2) and the roots of equation (6.8) are known as the characteristics or characteristic directions.

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\* If equations (6.8) cannot be satisfied by real roots characteristic directions do not exist and the system of differential equations is elliptic. If equation (6.8) has  $n/2$  real and distinct roots it is parabolic and if it has  $n$  real and distinct roots it is hyperbolic.



## SECTION 7

THE METHOD OF CHARACTERISTICS AND ITS APPLICATION  
TO SOLUTION OF THE T.L.E.s

In section 6) a matrix formulation of the Characteristic Equations of a set of hyperbolic P.D.E.s in two independent variables was developed. It is intended in this section, to show how these equations can be used to provide solutions of the T.L.E.s.

7.1) METHOD OF CHARACTERISTICS SOLUTION OF AN  $N^{\text{TH}}$  ORDER  
TRANSMISSION SYSTEM

An  $n^{\text{th}}$  order transmission system can be represented by the  $n$  hyperbolic P.D.E.s

$$[a] \frac{\partial}{\partial x} \{f\} + [b] \frac{\partial}{\partial t} \{f\} + [h] \{f\} = 0 \quad (7.1)$$

which in turn can be equivalently written as  $n$  sets of two ordinary differential equations (section 6.2)).

$$\frac{dt}{dx} = \xi_i \quad (7.2)$$

$$\{\lambda\}_i \{ [a] \frac{d}{dx} \{f\} + [h] \{f\} \} = 0 \quad i = 1, 2, \dots, n$$

The first equation of (7.2) defines a family of curves  $C_i$  in the  $(x, t)$  plane where at some point  $(x_1, t_1)$  the slope of the curve is given by:

$$\frac{dt}{dx} = \xi_i(x_1, t_1, \{f(x_1, t_2)\})$$

The second equation of (7.2) is an ordinary differential equation which applies only along the curve  $C_i$ . (See Fig.

(7.1)) In general  $\xi_i = \xi_i(x, t, \{f\})$ . However if the coefficients of equation (7.1) are independent of  $\{f\}$  the solution of the first and second equations of (7.2) separate simplifying their solution. Further, if in addition, the coefficients of equation (7.1) are independent of  $x$  and  $t$ , the  $\xi_i$  become constant and the curves  $C_i$  become straight lines in the  $(x, t)$  plane.

Consider an  $n^{\text{th}}$  order transmission system described by the T.L.E.s with constant coefficients. Suppose it is desired to solve for this system at some point  $(x_s, t_s)$ . For an  $n^{\text{th}}$  order system there will be  $n$  characteristic curves which are straight lines in the  $(x, t)$  plane. These are drawn to coincide at the point  $(x_s, t_s)$  and each is extended backwards in time until it reaches a point  $(x_i, t_i)$  at which initial values of  $\{f\}$  are known. (See Fig. (7.2)) Along each of the curves  $C_i$  applies one ordinary differential equation in  $\{f\}$ . Thus at point  $(x_s, t_s)$  there are  $n$  simultaneous differential equations in the  $n$  variables  $\{f\}$ . Each of these equations has a point  $(x_i, t_i)$  at which initial values of  $\{f\}$  are known. Integrating each of these differential equations from their initial points  $(x_i, t_i)$  along the curve  $C_i$  to the final point  $(x_s, t_s)$  enables the  $n$  unknowns  $\{f\}$  to be found at the point  $(x_s, t_s)$ . This method of solution is known as the Method of Characteristics.

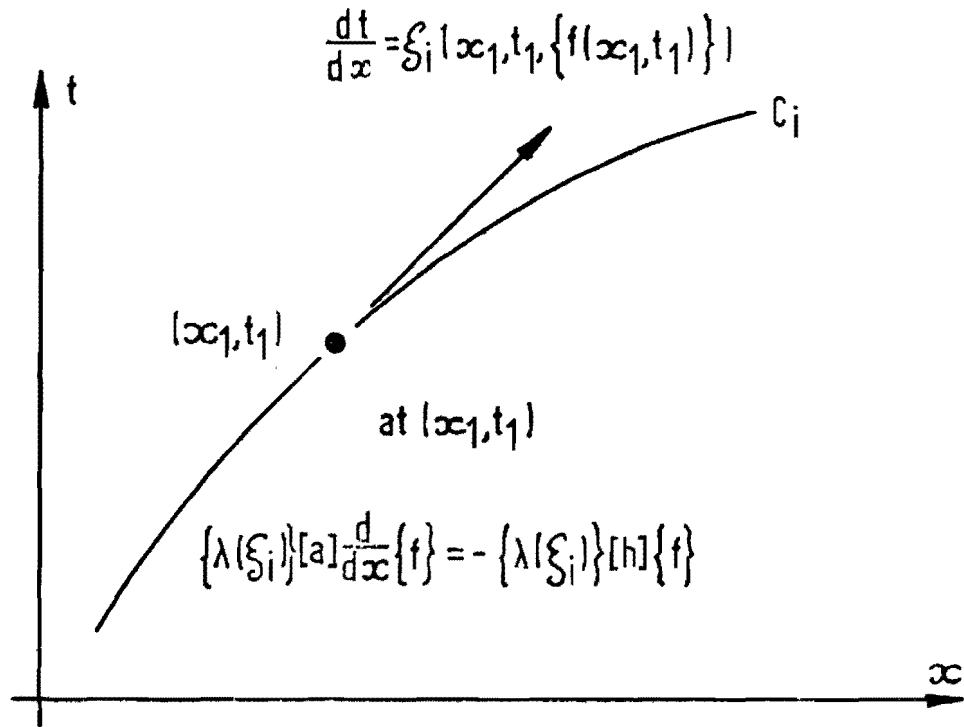
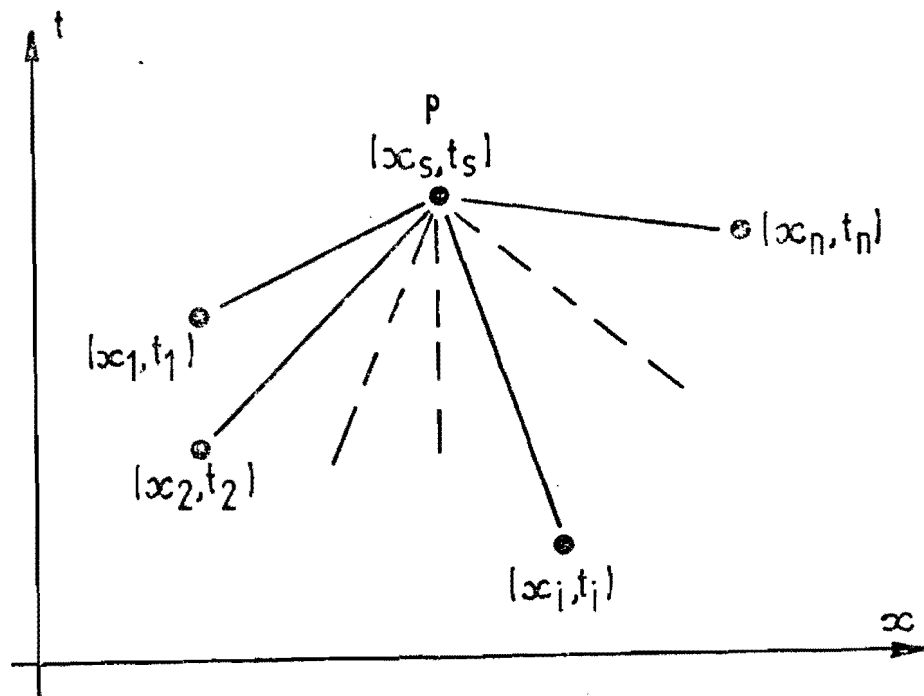


Fig. (7.1)

Fig. (7.2) Characteristic curves of an  $n^{\text{th}}$  order linear transmission system

## 7.2) CHARACTERISTIC EQUATIONS OF THE T.L.E.s

It is shown in section 6.2) that the characteristic directions,  $\xi_i$ , of an  $n^{\text{th}}$  order hyperbolic system are the roots of the quotient  $\frac{dt}{dx}$  in the  $n^{\text{th}}$  order polynomial:

$$|[b] - \frac{dt}{dx}[a]| = 0$$

For the T.L.E.s this equation becomes:

$$|[b] - \frac{dt}{dx}[I]| = 0 \quad (7.3)$$

where  $[I]$  is the unit matrix. Thus it is seen that these roots are the eigenvalues of the matrix  $[b]$ .

i.e.  $\xi_i$  are the eigenvalues of  $\begin{bmatrix} 0 & | & [L] \\ \hline [C] & | & 0 \end{bmatrix}$

The corresponding row vector  $\{\lambda(\xi_i)\}$  is the eigenvector obtained from:

$$[[b] - [I]\xi_i]^T \{\lambda\}^T = 0 \quad (7.4)$$

The associated ordinary differential equation becomes:

$$\{\lambda(\xi_i)\} \left\{ \frac{d}{dx} \{f\} + [h] \{f\} \right\} = 0 \quad (7.5)$$

along  $\frac{dt}{dx} = \xi_i$

We can thus construct the characteristic equations of the T.L.E.s by standard matrix operations on the matrix  $[b]$ , the T.L.E.s time derivative coefficient matrix. Since the required matrix operations should be found in any comprehensive matrix algorithm package the basis of a flexible computer construction of the T.L.E.s characteristic equations has been established.

### 7.3) CONDITIONS FOR AN EXACT SOLUTION OF THE T.L.E.s CHARACTERISTIC EQUATIONS

It will be useful to know under what conditions the characteristic equations of the T.L.E.s can be integrated analytically along their respective characteristic curves. If this is possible then exact solutions of the T.L.E.s can be obtained using the method of characteristics.

The differential equation applying along the characteristic  $\frac{dt}{dx} = \xi_i$  is:

$$\frac{d}{dx} \{ \{ \lambda(\xi_i) \} \{ f \} \} + \{ \lambda(\xi_i) \} [h] \{ f \} = 0$$

If this equation can be written in terms of a new single variable:

$$u = \{ \lambda(\xi_i) \} \{ f \} \quad (7.6)$$

then an exact solution can be obtained.

$$\text{Now} \quad \{ \lambda \} [h] \{ f \} = \sum_{k=1}^n \lambda_k \sum_{j=1}^n h_{kj} f_j$$

and the required condition is that this can be written in the form:

$$g \sum_{a=1}^n \lambda_a f_a = \sum_{k=1}^n \lambda_k \sum_{j=1}^n h_{kj} f_j$$

Equating the coefficients of  $f_a$  gives the condition:

$$\sum_{k=1}^n \frac{\lambda_k h_{k1}}{\lambda_1} = \sum_{k=1}^n \frac{\lambda_k h_{k2}}{\lambda_2} = \dots = \sum_{k=1}^n \frac{\lambda_k h_{kn}}{\lambda_n} \quad (7.7)$$

This defines a relationship between the coefficients of the loss terms and the eigenvectors. Note that an exact solution will always be obtained for a lossless line since for such a line  $[h] \equiv 0$ .

### 7.3.1) Application to a Single Circuit Transmission Line with Losses

For such a transmission line:

$$[h] = \begin{bmatrix} 0 & R \\ G & 0 \end{bmatrix} \quad (7.8)$$

The condition for an exact solution is:

$$\frac{\lambda_1 h_{11} + \lambda_2 h_{21}}{\lambda_1} = \frac{\lambda_1 h_{12} + \lambda_2 h_{22}}{\lambda_2}$$

From identity (7.8)  $h_{11} = h_{22} = 0$  so that the required condition becomes:

$$\frac{h_{21}}{h_{12}} = \left( \frac{\lambda_1}{\lambda_2} \right)^2 \quad (7.9)$$

From identity (7.8)  $\frac{h_{21}}{h_{12}} = \frac{G}{R}$

From equations (7.12)

$$\left( \frac{\lambda_1}{\lambda_2} \right)^2 = \frac{C}{L}$$

From which it is seen that an exact solution will be obtained if:

$$\frac{G}{R} = \frac{C}{L}$$

which defines a distortionless transmission line.

### 7.4) METHOD OF CHARACTERISTICS APPLIED TO A SINGLE CIRCUIT TRANSMISSION LINE

Writing the T.L.E.s in the form of equation (7.1) gives:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial \mathbf{x}} \begin{Bmatrix} \mathbf{v} \\ \mathbf{i} \end{Bmatrix} + \begin{bmatrix} 0 & L \\ C & 0 \end{bmatrix} \frac{\partial}{\partial t} \begin{Bmatrix} \mathbf{v} \\ \mathbf{i} \end{Bmatrix} + \begin{bmatrix} 0 & R \\ G & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{v} \\ \mathbf{i} \end{Bmatrix} = 0$$

The characteristic directions are the eigenvalues of matrix [b] and so are found from:

$$\begin{vmatrix} -\xi & L \\ C & -\xi \end{vmatrix} = 0$$

which gives  $\xi = \pm\sqrt{LC}$  (7.10)

Thus the first of equations (7.2) are:

$$\frac{dt}{dx} = +\sqrt{LC} \quad \text{and} \quad \frac{dt}{dx} = -\sqrt{LC}$$

The corresponding eigenvectors are the  $\{\lambda\}$  which satisfy the equations:

$$\begin{bmatrix} -\xi & C \\ L & -\xi \end{bmatrix} \begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix} = 0 \quad (7.11)$$

For  $\xi = +\sqrt{LC}$  we obtain:

$$-\sqrt{LC} \lambda_1 + C\lambda_2 = 0 \quad \text{or} \quad \left( \frac{\lambda_1}{\lambda_2} \right)^{-1} = +\sqrt{\frac{L}{C}} \quad (7.12)$$

For  $\xi = -\sqrt{LC}$  we obtain  $\left( \frac{\lambda_1}{\lambda_2} \right)^{-1} = -\sqrt{\frac{L}{C}}$

so that the eigenvectors are:

$$\{1, \sqrt{\frac{L}{C}}\} \lambda_1 \quad \text{for} \quad \xi = +\sqrt{LC}$$

$$\{1, -\sqrt{\frac{L}{C}}\} \lambda_1 \quad \text{for} \quad \xi = -\sqrt{LC}$$

where  $\lambda_1$  remains an undetermined multiplier.

Thus the second of equations (7.2) are for  $\xi = +\sqrt{LC}$

$$\lambda_1 \{1, \sqrt{\frac{L}{C}}\} \left\{ \frac{d}{dx} \begin{Bmatrix} v \\ i \end{Bmatrix} + \begin{bmatrix} 0 & R \\ G & 0 \end{bmatrix} \begin{Bmatrix} v \\ i \end{Bmatrix} \right\} = 0$$

$$\text{along } \frac{dt}{dx} = +\sqrt{LC}$$

which when expanded reduces to:

$$\frac{d}{dx} (v + \sqrt{\frac{L}{C}} i) = -Ri - G\sqrt{\frac{L}{C}} v$$

$$\text{along } \frac{dt}{dx} = +\sqrt{LC}$$

$$\text{and similarly for } \xi = -\sqrt{LC} \quad (7.13)$$

$$\frac{d}{dx} (v - \sqrt{\frac{L}{C}} i) = -Ri + G\sqrt{\frac{L}{C}} v$$

$$\text{along } \frac{dt}{dx} = -\sqrt{LC}$$

The equivalent equations with respect to  $t$ , obtained by changing the variable differentiation are:

$$\frac{d}{dt} (\sqrt{LC} v + Li) = -Ri - G\sqrt{\frac{L}{C}} v$$

$$\text{along } \frac{dt}{dx} = +\sqrt{LC}$$

(7.14)

$$\frac{d}{dt} (\sqrt{LC} v - Li) = Ri - G\sqrt{\frac{L}{C}} v$$

$$\text{along } \frac{dt}{dx} = -\sqrt{LC}$$

Equations (7.13) and (7.14) are the characteristic equations of a single circuit transmission line. In them  $\frac{dt}{dx} = +\sqrt{LC}$  is known as the forward characteristic and  $\frac{dt}{dx} = -\sqrt{LC}$  is known as the backward or reverse characteristic. These two characteristics define a domain of influence in the  $(x, t)$  plane for a signal applied at some point on the line.



In Fig. (7.3) a time varying signal is applied at point  $x_0$  on a line at time  $t = 0$ . Since the equations describing propagation of this signal are constrained to apply only in these characteristic directions there is no way in which the unshaded area can be affected by the disturbance. The characteristic curves are thus the loci of disturbances or waves propagating through the  $(x,t)$  plane.

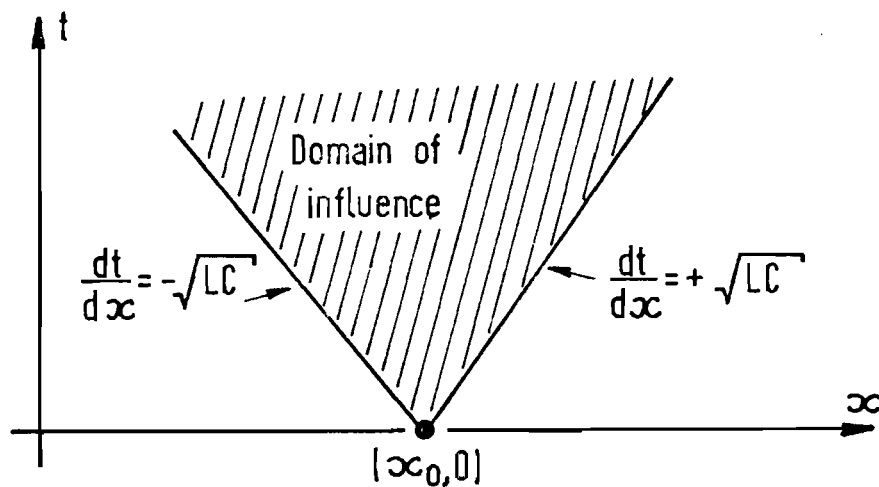


Fig. (7.3) Disturbance propagation in  $(x,t)$  plane

Along the forward characteristic, in terms of the coordinates, a forward travelling wave is characterized by both the voltage and current having the same sign. Along the reverse characteristic a backward propagating wave is characterized by the voltage and current having opposite signs. Note that the two equations (7.14) can be interchanged by changing the sign of the current. This symmetry in the characteristic equations is expected since the mechanism of propagation is the same whether it is in the forward or the reverse direction.

#### 7.4.1) Solution for a Lossless Transmission Line

For this hypothetical transmission line there are no loss terms and the corresponding characteristic equations reduce to:

$$\frac{d}{dx} (v + \sqrt{\frac{L}{C}} i) = 0 \quad \text{along} \quad \frac{dt}{dx} = +\sqrt{LC}$$

$$\frac{d}{dx} (v - \sqrt{\frac{L}{C}} i) = 0 \quad \text{along} \quad \frac{dt}{dx} = -\sqrt{LC}$$

Integrating these with respect to  $x$  along their respective characteristic curves gives:

$$(v + \sqrt{\frac{L}{C}} i)_{(x_s, t_s)} = (v + \sqrt{\frac{L}{C}} i)_{(x_1, t_1)} \quad (7.15)$$

$$(v - \sqrt{\frac{L}{C}} i)_{(x_s, t_s)} = (v - \sqrt{\frac{L}{C}} i)_{(x_2, t_2)}$$

where the points  $(x_s, t_s)$ ,  $(x_1, t_1)$  and  $(x_2, t_2)$  are as defined in Fig. (7.4).

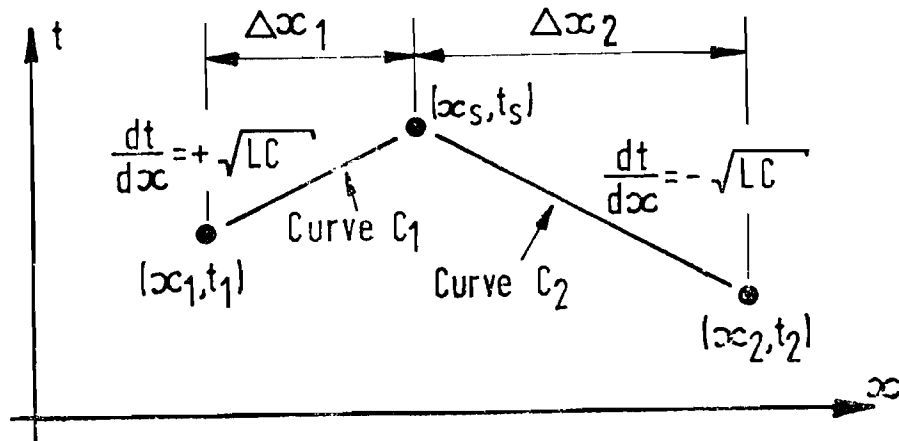


Fig. (7.4) Coordinate definition for solution of the T.L.E.s characteristic equations.

Equations (7.15) can be used to solve for the point on the line  $x_s$  at time  $t_s$ . Mostly however, the characteristic equations are integrated between the line boundaries according to Fig. (4.4) giving a solution for propagation between the line boundaries directly. (See Section 4.4.3))

#### 7.4.2) Solution for a Distortionless Transmission Line

For a distortionless line the T.L.E.s contain the effects of both series resistance and shunt conductance according to the relation:

$$\frac{R}{L} = \frac{G}{C}$$

Substituting  $\frac{RC}{L}$  for  $G$  in equation (7.13) gives:

$$\begin{aligned} \frac{d}{dx} (v + Z_c i) &= -\frac{R}{Z_c} (v + Z_c i) \\ \text{along } \frac{dt}{dx} &= +\sqrt{LC} \end{aligned} \quad (7.16)$$

$$\begin{aligned} \frac{d}{dx} (v - Z_c i) &= +\frac{R}{Z_c} (v - Z_c i) \\ \text{along } \frac{dt}{dx} &= -\sqrt{LC} \end{aligned}$$

where  $Z_c = \sqrt{\frac{L}{C}}$

Integrating equations (7.16) according to Fig. (7.4) gives:

$$\begin{aligned} (v + Z_c i)_{(x_s, t_s)} &= (v + Z_c i)_{(x_1, t_1)} e^{-\frac{R|\Delta x_1|}{Z_c}} \quad \text{along } C_1 \\ (v - Z_c i)_{(x_s, t_s)} &= (v - Z_c i)_{(x_2, t_2)} e^{-\frac{R|\Delta x_2|}{Z_c}} \quad \text{along } C_2 \end{aligned} \quad (7.17)$$

Alternatively equations (7.16) can be integrated between the line boundaries according to Fig. (4.4) (section 4.4.3)) to give:

$$\begin{aligned} (v + Z_c i)_{x_1} &= (v + Z_c i)_{x_0} e^{-\frac{Rd}{Z_c}} \quad \text{along } C_1 \\ (v - Z_c i)_{x_0} &= (v - Z_c i)_{x_1} e^{-\frac{Rd}{Z_c}} \quad \text{along } C_2 \end{aligned} \quad (7.18)$$

where  $d$  is the length of the transmission line.

Equations (7.17) and (7.18) can be reduced to the usual expression for a distortionless transmission line by substituting  $v = Z_c i$ .

Equations (7.18) will be used to develop a graphical solution for a distortionless line in section 9).

#### 7.4.3) Solution for General Lossy Transmission Line

Recalling the comments on equations (7.14) immediately before section 7.4.1) it will be seen that a backward propagating wave contributes nothing to the value of  $(v + Z_c i)$  used in equations (7.17) and (7.18). This means that for a distortionless line, forward propagating waves are handled entirely by solving the equations applying along the forward characteristics. A similar statement can be made for backward propagating waves. This observation can also be made for a lossless transmission line. Unfortunately the same cannot be said of the general lossy transmission line where there is no longer a constant relation between the voltage and current. The characteristic equations cannot be written in terms of a single variable and the solutions for both forward and backward propagating waves become dependent

on the equations applying along both forward and reverse characteristics simultaneously.

Integrating equations (7.13) along their respective characteristic curves  $C_1$  and  $C_2$  in accordance with Fig. (7.4) gives:

$$\begin{aligned}
 (v + Z_c i)_{(x_s, t_s)} - (v + Z_c i)_{(x_1, t_1)} &= - \int_{x_1}^{x_s} R dx - \int_{x_1}^{x_s} G Z_c v dx \\
 &\quad \text{along } C_1 \quad \text{along } C_1 \\
 (v - Z_c i)_{(x_s, t_s)} - (v - Z_c i)_{(x_2, t_2)} &= - \int_{x_2}^{x_s} R dx + \int_{x_2}^{x_s} G Z_c v dx \\
 &\quad \text{along } C_2 \quad \text{along } C_2
 \end{aligned}
 \tag{7.19}$$

These two equations describe wave propagation for the general lossy line. However, the integrals require the functional forms of  $v$  and  $i$  to be known along the characteristic curves. In general these will not be known and exact solutions cannot be obtained. Approximate solutions can be obtained by assuming functions for  $v$  and  $i$ . The extent of the approximation depends on the functions that are assumed. This is examined in detail in section 10.2) where a number of different function approximations are considered.

## SECTION 8

### CONSTANT PARAMETER POWER SYSTEM TRANSMISSION LINE MODEL

The constant parameter transmission line is frequently used in power systems analysis. In this transmission line the T.L.E.s are used directly and without modification. There are four variants of this model in common use.

- 1) Lossless model
- 2) Distortionless model
- 3) Low loss model
- 4) Series resistance model.

These models are all approximate in their representation of power system transmission lines. They can, however, be used; subject to their limitations and the degree of approximation that can be tolerated.

#### 8.1) ASSUMPTIONS

Consider the typical power system transmission line shown in Fig. (8.1). Use of the constant parameter model assumes that we have a transmission line such that:

- (a) The line conductors miraculously levitate at a constant height above the ground. This neglects
  - 1) conductor sag,
  - 2) lumped capacitive loading.
- (b) The air is a perfect dielectric. This neglects corona discharge by ionized air.

- (c) The earth is a perfectly conducting plane. This neglects diffusion of currents into the ground.
- (d) The line is of infinite extent. This neglects terminal effects thus representing the transmission phenomena correctly only at some distance from points of discontinuity.
- (e) The conductor losses and internal inductance are constant. This neglects skin effect in conductors.
- (f) No radiation.

In addition: Series resistance model - makes no more assumptions.

Low loss model - limits loss values.

Distortionless model - fabricates shunt conductance.

Lossless model - neglects line losses.

Of the above assumptions, (d) is shown by CARSON (1928) to be of negligible order. Assumptions (c) and (e) concern frequency dependent line parameters which can introduce significant errors if not accounted for, especially if the ground forms part of the return circuit. These will be dealt with in more detail in sections 15) and 16) of this thesis. Assumption (b) is true provided that the transmission line voltages are below the critical breakdown levels. If the voltages are above these levels the results can be significantly in error. Assumption (a)-1) is examined in detail by PEREL'MAN (1968). It is shown that the propagation

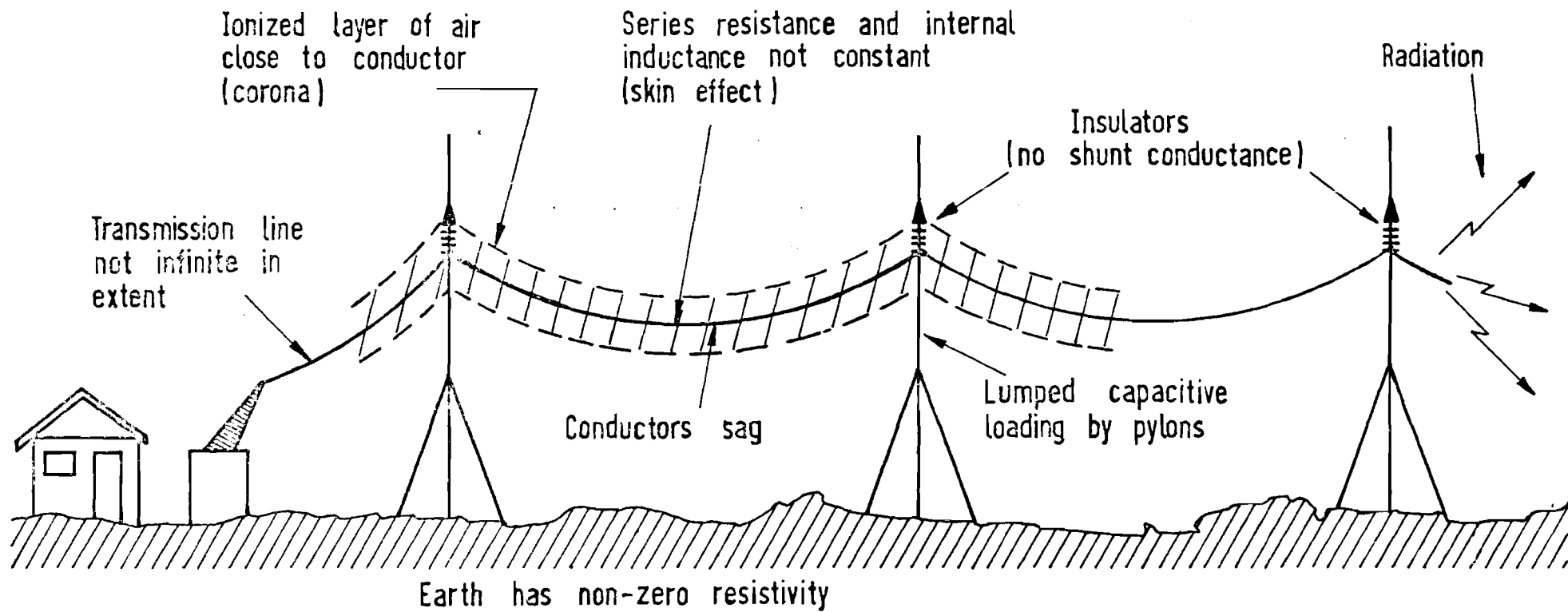


Fig. (8.1) Typical power system transmission line.



constant and the characteristic impedance will be in error by less than 1 percent apart from a small band of frequencies which are characterised in that the length of the line span is approximately a multiple of the half wavelength. For these resonant frequencies the attenuation factor is considerably increased. This selectively attenuates a small number of frequencies and would not significantly affect surge propagation. Assumption (a)-2) is examined by BEWLEY (1963). The lumped capacitive loading is shown to have an insignificant effect on the slope of the wavefront. However, the effective velocity of propagation is decreased. These properties can be closely approximated by slightly increasing the capacitance of the line thus maintaining a constant parameter model. Assumption (f) will have negligible effect on surge propagation. At very high frequencies displacement currents may prevent a perfect open circuit at the open end of a line. However, the effect of this is unlikely to be greater than the end effects already neglected.

From the above remarks it will be evident that only assumptions (c), (e) and sometimes (b) will cause significant differences between surge propagation predicted by a constant parameter transmission line and that observed on a typical power system transmission line. Assumptions (c) and (e) will affect predominantly the wavefronts and (b) the shape of high voltage signals. For signals below the critical corona levels and where there is no earth return path the constant parameter model can give acceptable results for both short and long term transients.

## 8.2) PROPAGATION CHARACTERISTICS

### 8.2.1) Series Resistance Model

Of the four variants of the constant parameter line only the series resistance line requires no extra assumptions additional to (a) - (f). It will thus be the most accurate of the four representations and hence will be used as the standard against which the others are compared. CARSON'S solution (section 4.6.1)) of a surge propagating along a line will be used to compute the results. The equations used are (4.23) in which  $G$  has been set to zero. The resulting solution is shown in Fig. (4.5). From this solution HEDMAN (1971) lists the following properties of propagation.

- " 1) The front of the wave propagates with velocity 'a'.
- 2) The current and voltage waves are dissimilar.
- 3) At the arrival of the toe of this wave, the current and voltage are related by the surge impedance,  $Z_c$ , but for subsequent time the ratio changes.
- 4) After the arrival of the wavefront, the voltage rises monotonically with time, but the time constant is long relative to the travel time of a normal power transmission line.
- 5) The effective transmission line input impedance (defined as source voltage/input current) increases with time for a unit step on an infinite line. "

To these observations can also be added (CARSON, 1926)

- " 6) At a point  $x$  on the line, the current is zero until the time  $t = \frac{x}{a}$  at which time it jumps to the value  $(1/Z_c)e^{-\rho \frac{x}{a}}$ . It then begins to die away provided

$\frac{\rho x}{a} < 2$ . If, however,  $\frac{\rho x}{a} > 2$ , the current begins to rise and may attain a maximum value very large compared with the head before it begins to die away.

- 7) When a sufficient time has elapsed after the arrival of the head of the wave, the waves become closer and closer to the waves of the corresponding non-inductive cable. "

From statement 7) CARSON concluded that the line inductance plays no part in the subsidence of the waves to their final values.

#### 8.2.2) Lossless Model

Here  $R = G = 0$  and both the voltage and current propagate unattenuated and undistorted according to:

$$v = v_0 \left( t - \frac{x}{a} \right)$$

$$i = i_0 \left( t - \frac{x}{a} \right)$$

Although this model is frequently used in power systems analyses, especially in graphical solutions of surge problems (see section 4.4.4)) it is essentially unsatisfactory for all but the briefest examination of transient problems (Section 8.2.5)).

#### 8.2.3) Distortionless Model

This model is also frequently used in power systems analysis, especially since the exponential attenuation is easily represented in a computer. It can also be used for the graphical solution of surge problems (Section 9)). For this model:

$$\frac{L}{R} = \frac{C}{G}$$

and

$$v = v_0 \left(t - \frac{x}{a}\right) e^{-\frac{\rho x}{a}}$$

$$i = i_0 \left(t - \frac{x}{a}\right) e^{-\frac{\rho x}{a}} = v/z_c$$

Like the lossless model this model propagates without distortion. The attenuation is doubled in comparison with the series resistance model.

#### 8.2.4) Low Loss Model (with $G = 0$ )

This line does not appear to be much used in power systems analysis. It is essentially a high frequency approximation to the series resistance line and has a response (APPENDIX (8.1)):

$$v = v_0 \left(t - \frac{x}{a}\right) e^{-\frac{\rho x}{a}}$$

$$i = \frac{1}{z_c} \left\{ v_0 \left(t - \frac{x}{a}\right) e^{-\frac{\rho x}{a}} - \rho e^{-\frac{\rho x}{a}} \int_{\frac{x}{a}}^t v \left(t - \frac{x}{a}\right) dt \right\} \quad (8.1)$$

from which it is seen that  $i$  at  $x = 0$  for a unit impressed step of voltage is a linear function of time.

$$i = (1 - \frac{R}{2L} t) / z_c$$

This line has the same attenuation as the series resistance line. The voltage solution is distortionless but the current has an extra integral term. Thus the voltage and the current do not maintain a constant relation.

#### 8.2.5) Comparison of Line Models

The responses of the different transmission lines are given in Fig. (8.2). Consider first the propagated wave (Fig. (8.2)(b)). It is seen that at the toe of the wave neither the lossless line nor the distortionless line have the same attenuation as the series resistance line. The distortionless line can, however, be made to have the same attenuation by halving the line series resistance. This, shown as the modified distortionless line, is the form in which such an approximation would be used in an analysis. It will be observed that only the series resistance line distorts the voltage step. However, for short periods of time after the wavefront has passed, the low loss line and the modified distortionless line could satisfactorily be used to represent a series resistance line.

Considering the line terminal current, it is seen that neither the lossless nor the distortionless lines satisfactorily represent the terminal loading of equipment by a series resistance line. The low loss line gives a good approximation for short periods of time after the voltage step is applied but departs markedly from the series resistance response for long periods of time.

Simple representations can be used to represent a series resistance line for short periods of time. However, where a response over both short and long periods of time is required the series resistance line must be used.

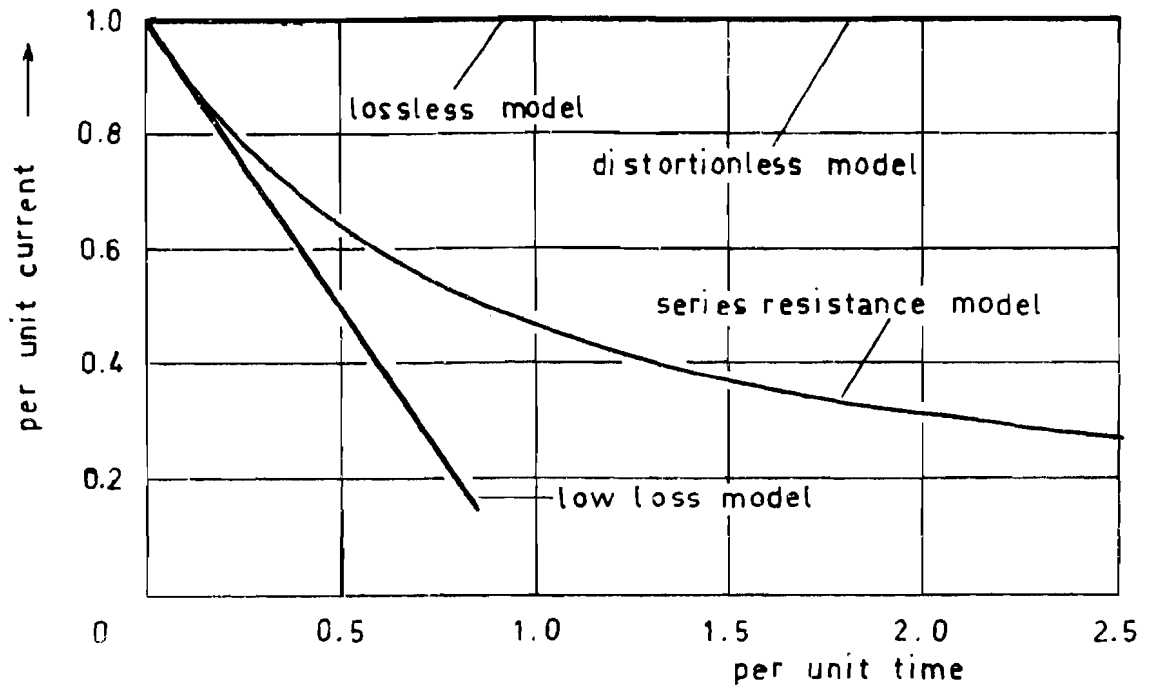


Fig. (8.2) (a) line input current — comparison of solutions for unit step applied voltage

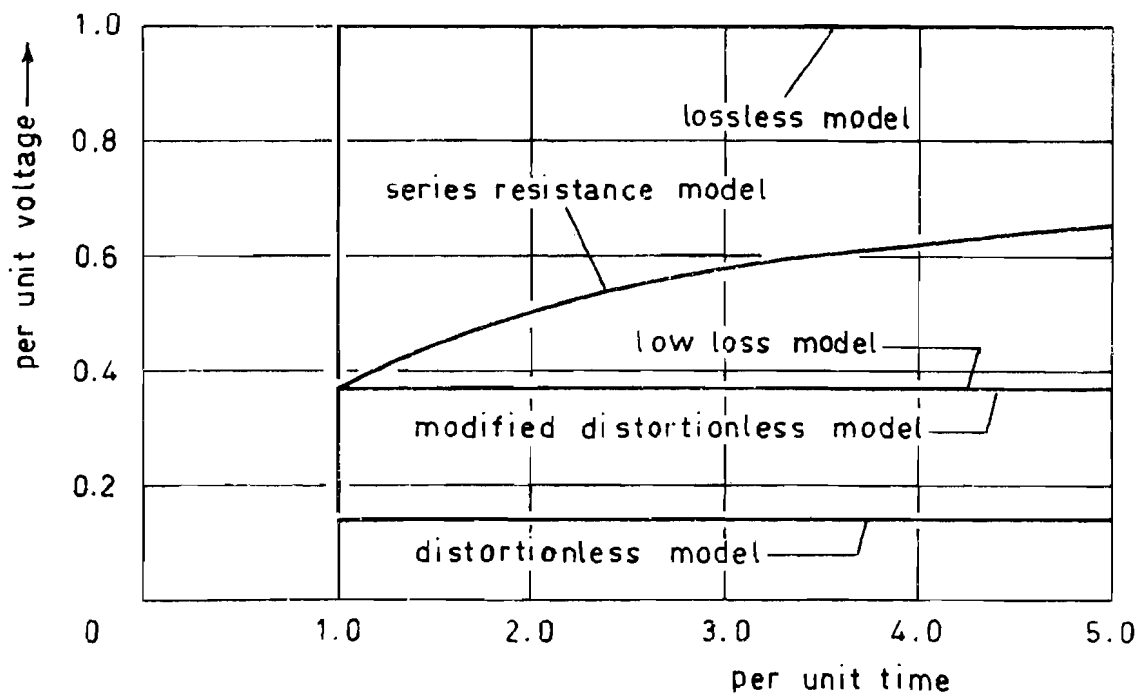


Fig. (8.2) (b) line travelling voltage wave — comparison of solutions for unit step applied voltage

## APPENDIX (8.1)

DERIVATION OF SOLUTION FOR LOW LOSS LINE

(High frequency approximation to general line)

$$- \frac{\partial v}{\partial x} = L \frac{\partial i}{\partial t} + Ri$$

$$- \frac{\partial i}{\partial x} = C \frac{\partial v}{\partial t} + Gv$$

Taking Laplace Transforms w.r.t. time for a line initially at rest:

$$\begin{aligned} - \frac{dV(x,s)}{dx} &= (sL + R) I(x,s) \\ - \frac{dI(x,s)}{dx} &= (sC + G) V(x,s) \end{aligned} \quad (9.2)$$

or, eliminating  $I(x,s)$  and  $V(x,s)$

$$\begin{aligned} \frac{d^2 V(x,s)}{dx^2} &= (sL + R)(sC + G) V(x,s) \\ \frac{d^2 I(x,s)}{dx^2} &= (sL + R)(sC + G) I(x,s) \end{aligned} \quad (9.3)$$

For a forward propagating wave equations (9.3) have the solution:

$$\begin{aligned} v(x,s) &= Ae^{-\gamma x} \\ I(x,s) &= \frac{\gamma}{(sL+R)} V(x,s) \end{aligned} \quad (9.4)$$

where  $\gamma = \{(sL+R)(sC+G)\}^{\frac{1}{2}}$  and  $A = V(0,s) = V_0(s)$ .

Now  $\gamma = s\sqrt{LC} \left(1 + \frac{1}{s} \left(\frac{R}{L} + \frac{G}{C}\right) + \frac{RG}{sLC}\right)^{\frac{1}{2}}$

If  $R$  and  $G$  are very small or alternatively  $sC \gg G$ ,  $sL \gg R$  (high frequency approximation) then expanding binomially:

$$\gamma \approx s\sqrt{LC} \left(1 + \frac{1}{2s} \left(\frac{R}{L} + \frac{G}{C}\right)\right)$$

so that:

$$V(x,s) = V_0(s) e^{-xs\sqrt{LC}} e^{-\frac{\sqrt{LC}}{2} \left(\frac{R}{L} + \frac{G}{C}\right) x}$$

Taking inverse transforms gives:

$$v(x,t) = v_0\left(t - \frac{x}{a}\right) e^{-\frac{\rho x}{a}}$$


---

$$\text{where } a = \frac{1}{\sqrt{LC}} \quad \text{and} \quad \rho = \frac{1}{2} \left(\frac{R}{L} + \frac{G}{C}\right)$$

$$I(x,s) = \frac{\gamma}{(sL+R)} V(x,s) = \sqrt{\frac{C}{L}} \left(1 + \frac{G}{sC}\right)^{\frac{1}{2}} \left(1 + \frac{R}{sL}\right)^{-\frac{1}{2}} V(x,s)$$

Again, expanding binomially and neglecting second and higher order terms gives:

$$I(x,s) = \frac{1}{Z_c} \left(1 - \frac{\sigma}{s}\right) V(x,s)$$

$$\text{where } Z_c = \sqrt{\frac{L}{C}} \quad \text{and} \quad \sigma = \frac{1}{2} \left(\frac{R}{L} - \frac{G}{C}\right)$$

Taking inverse transforms gives:

$$i(x,t) = \frac{1}{Z_c} \left\{ v_0\left(t - \frac{x}{a}\right) e^{-\frac{\rho x}{a}} - \sigma e^{-\frac{\rho x}{a}} \int_{\frac{x}{a}}^t v\left(T - \frac{x}{a}\right) dT \right\}$$


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## SECTION 9

A GRAPHICAL METHOD OF SOLUTION FOR TRAVELLING WAVES ON  
TRANSMISSION SYSTEMS WITH ATTENUATION

Graphical methods of solution provide a feasible means of examining smaller travelling wave problems. They have the advantage that no computing facilities are required and that pictorial results are obtained directly. As solution is in the  $(v,i)$  plane, non-linear line terminations can be handled with ease.

The traditional graphical solution method (ARLETT, 1966) (BERGERON, 1961) uses the characteristic equations of a lossless transmission line (equations (4.16) of section 4.4.3)) and is discussed briefly in section 4.4.4). However, use of the lossless transmission line has significant shortcomings. Transient phenomena being unattenuated can persist indefinitely. The overvoltages calculated can be significantly higher than will actually obtain. The distortion associated with typical power system transmission lines is neglected.

A better approximation is to use a distortionless but lossy transmission line. This enables attenuation to be taken into account while at the same time preserving a simple form of solution. Distortion is still not accounted for but in the region of the wavefront attenuation is the dominant property of propagation. (See section 8.2.5)). Such an approximation will yield good results for small periods of time after the wavefront has passed. Graphical methods of solution tend to be used for problems involving shorter

rather than longer time periods as for longer time periods they tend to become cluttered. Thus a graphical solution in which attenuation can be included will allow a more realistic appraisal of transient problems associated with transmission systems whose attenuation cannot be neglected. Such a graphical method of solution will now be developed.

### 9.1) GRAPHICAL CONSTRUCTION OF TRANSMISSION EQUATIONS

Equations (7.18) derived in section 7.4.2) will be used in this solution. These, describing propagation along a distortionless transmission line, are given below.

$$\begin{aligned} (v + Z_c i)_{x_1} &= (v + Z_c i)_{x_0} e^{-R_L d / Z_c} \quad \text{along } C_1 \\ (v - Z_c i)_{x_0} &= (v - Z_c i)_{x_1} e^{-R_L d / Z_c} \quad \text{along } C_2 \end{aligned} \quad (9.1)$$

where the subscripts are in accordance with Fig. (9.1),

$d$  is the length of the transmission line and

$R_L$  is the transmission line resistance per unit length. Alternatively equations (9.1) can be written in terms of the transit time ' $T$ ' of the transmission line by substituting  $R_L T / L$  for  $R_L d / Z_c$  where  $L$  is the line inductance per unit length.

A signal propagating from end  $x_0$  to end  $x_1$  of a distortionless line is described by the equation:

$$(v + Z_c i)_{x_1} = K \quad (9.2)$$

where  $K$  is a constant equal to  $(v + Z_c i)_{x_0} e^{-R_L T / L}$

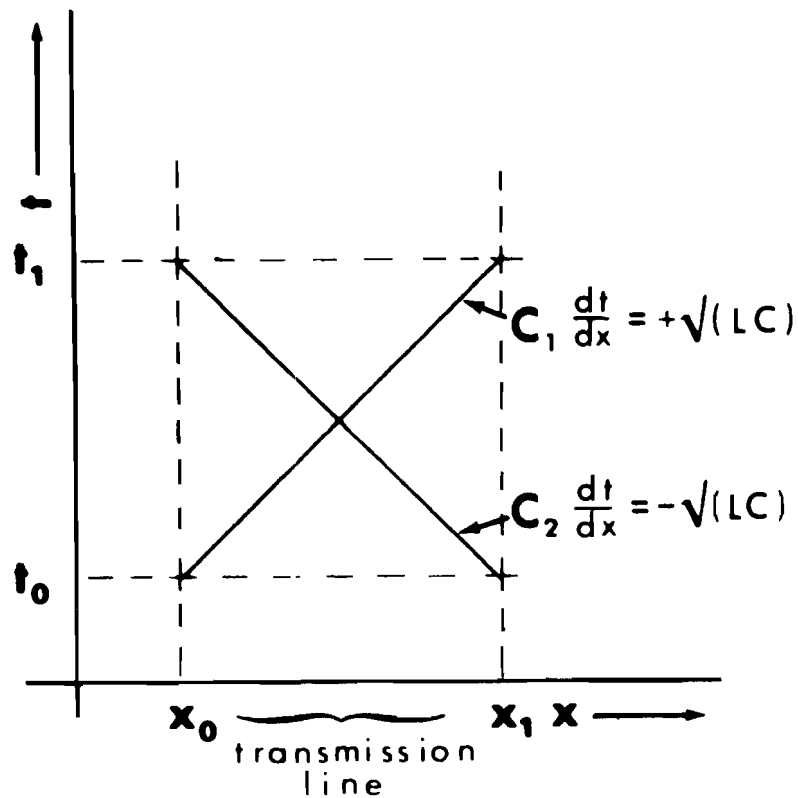


Fig. (9.1) Subscript and characteristic curves convention used in equations (9.1)

This, as shown in Fig. (9.2) is a straight line of slope  $-Z_c$  passing through the point  $(v_{x_0\beta}, i_{x_0\beta})$  in the  $(v, i)$  plane. Here  $\beta = e^{-R_L T/L}$  and  $(v_{x_0}, i_{x_0})$  is the value of voltage and current at end  $x_0$  at a time  $T$  before the signal arrives at end  $x_1$ . To construct this line the point  $(v_{x_0\beta}, i_{x_0\beta})$  must be found from the point  $(v_{x_0}, i_{x_0})$ . Since both  $v$  and  $i$  are reduced by the same amount the point  $(v_{x_0\beta}, i_{x_0\beta})$  will lie somewhere on the straight line drawn from point  $(v_{x_0}, i_{x_0})$  to the origin. (See Fig. (9.3)). Thus only  $v_{x_0\beta}$  or  $i_{x_0\beta}$  need to be determined initially. Suppose that  $v_{x_0\beta}$  is to be determined. A number of graphical constructions are possible. In one, a straight line is drawn from the origin

with a slope  $\frac{1}{\beta}$ . Given  $v = v_{x_0}$ ,  $v_{x_0} \beta$  is read off directly and reflected back onto the  $V$  axis. This is shown in Fig. (9.3). Alternatively a straight line with a slope  $-\frac{1}{\beta}$  is drawn from  $v = v_{x_0}$  to cut the horizontal axis at  $v_{x_0} \beta$  which is then reflected back onto the  $V$  axis. This is shown in Fig. (9.4). Where a line terminal relation is present, such as  $v = E - ir$  for example, it may be more convenient to draw  $v = (E - ir) \beta$  underneath it and obtain  $v_{x_0} \beta$  by dropping a line vertically to intersect  $v = (E - ir) \beta$  at  $v_{x_0} \beta$ . This is shown in Fig. (9.5). Given  $v_{x_0} \beta$ ,  $i_{x_0} \beta$  is found from the intersection of the line  $v = v_{x_0} \beta$  and the line from  $(v_{x_0}, i_{x_0})$  to the origin. The equation of forward propagation can then be drawn.

Backward propagation is handled in a similar manner and is a line of slope  $+Z_c$  passing through the point  $(v_{x_1} \beta, i_{x_1} \beta)$  in the  $(v, i)$  plane. (See Fig. (9.2))

## 9.2) GRAPHICAL SOLUTION OF TRAVELLING WAVE PROBLEMS

The graphical solution of networks containing distortionless transmission lines is achieved firstly by drawing the  $v - i$  relations at each line terminal and then finding the simultaneous solution of these and the equations of propagation by noting their points of intersection.

To illustrate the graphical solution procedure five examples will be given. In the first we solve for a line being charged from a d.c. source of infinite capacity.

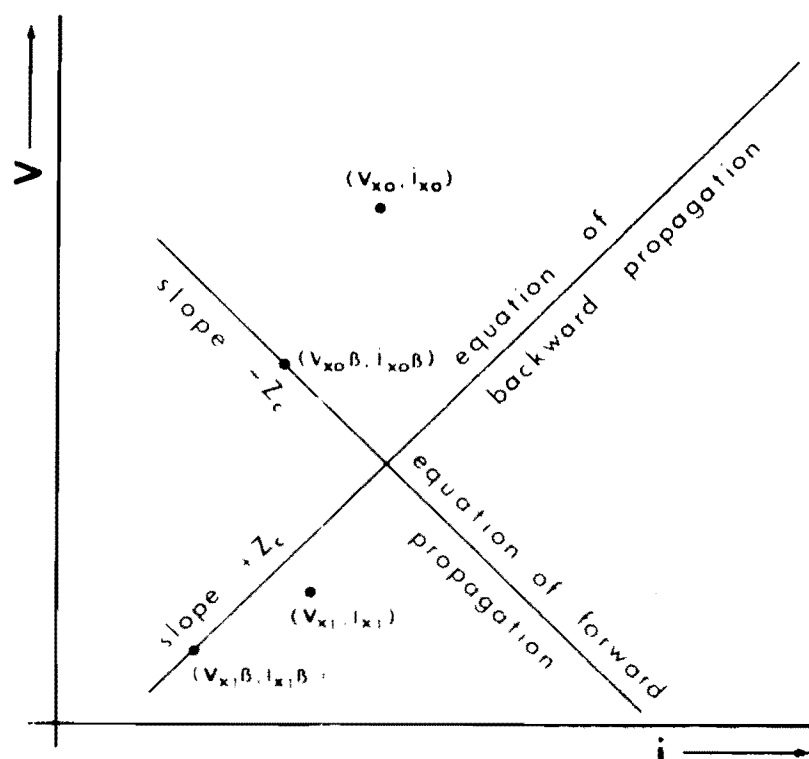
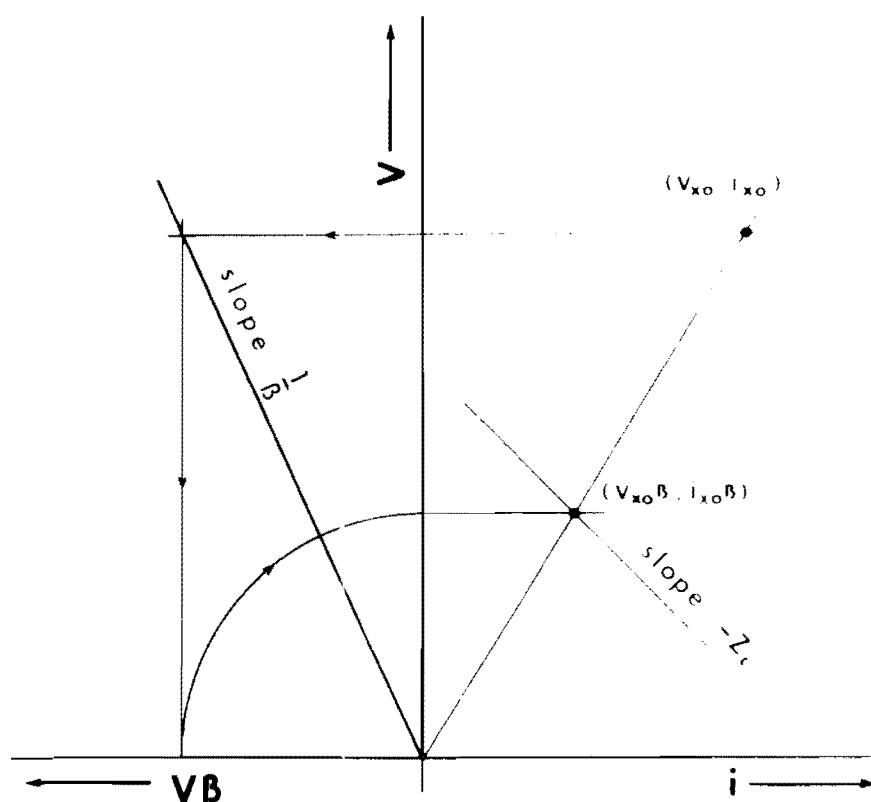


Fig. (9.2) Equations of propagation for a distortionless transmission line



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Fig. (9.3) Graphical construction for equation of forward propagation.



### 9.2.1) Charging of a Line from a D-C Source

This consists of applying a step of voltage to one end of a line initially at rest with the other end open circuited. The procedure is given in Fig. (9.6). The equation at end  $x_0$  is  $v = E$ . The equation at end  $x_1$  is  $i = 0$ , or the  $V$  axis. At the instant that the voltage  $E$  is applied to  $x_0$ , two equations are simultaneously valid.

$$\begin{aligned} v_{x_0} &= E \\ (v - Z_c i)_{x_0} &= (v - Z_c i)_{x_1} \beta \end{aligned} \tag{9.3}$$

The second of equations (9.3) is the equation of backward propagation and takes account of initial conditions on the line. (In this case a signal applied to end  $x_1$  at a time  $T$  before the voltage  $E$  is applied to end  $x_0$ .) However, since the line is initially at rest the right hand side is zero and the second of equations (9.3) becomes:

$$(v - Z_c i)_{x_0} = 0$$

The solution of equations (9.3) is at point '0', so labelled because zero line transit times have elapsed since the signal was applied. Point '0' gives the voltage and current on the line at end  $x_0$  the instant that the voltage step is applied.  $(v_{x_0} \beta, i_{x_0} \beta)$  at this time is given by point '0a'. From point '0a' the equation of forward propagation is drawn. This is a straight line with slope  $-Z_c$  through point '0a'. The intersection of this line with the  $V$  axis at point '1' gives the solution at end  $x_1$ , one transit time after the signal was applied. Point '1a' gives  $(v_{x_1} \beta, i_{x_1} \beta)$  at that time. From point '1a' the equation of backward propagation

is drawn to intersect  $V = E$  at point '2' giving the solution at end  $x_0$  two transit times after the signal was applied. Point '2a' giving  $(v_{x_0}, i_{x_0})$  at that time is found by drawing a straight line from point '2' to the origin and finding where it intersects  $V = E\beta$ .

The final steady state voltage at the open circuit end can be shown to be (BEWLEY, 1963):

$$v_{oc}^{\infty} = \frac{2\beta}{1+\beta^2} E$$

In this example  $\beta$  was chosen to be 0.5 giving:

$$v_{oc}^{\infty} = 0.8E.$$

### 9.2.2) A Step of Voltage Applied to a Line Terminated in a Short Circuit

In Fig. (9.7) the construction has been carried out for the same transmission line but terminated at its far end by a short circuit instead of an open circuit. Here the equation at the far end is  $V = 0$ , or the  $i$  axis. This necessitates calculating  $i\beta$  at end  $x_1$  instead of  $V\beta$  as in the previous example.

### 9.2.3) A Step of Voltage Applied to a Line Terminated in a Resistance 'R'

#### (a) Case for $R < Z_c$

The construction required in this case is given in Fig. (9.8). For this transmission line  $\beta = 0.8$  has been chosen. The equation at end  $x_1$  is  $v_R = i_R R$ . As before point '0' gives the voltage and current at  $x_0$  the instant that the voltage is applied. From point '0a' the equation of forward



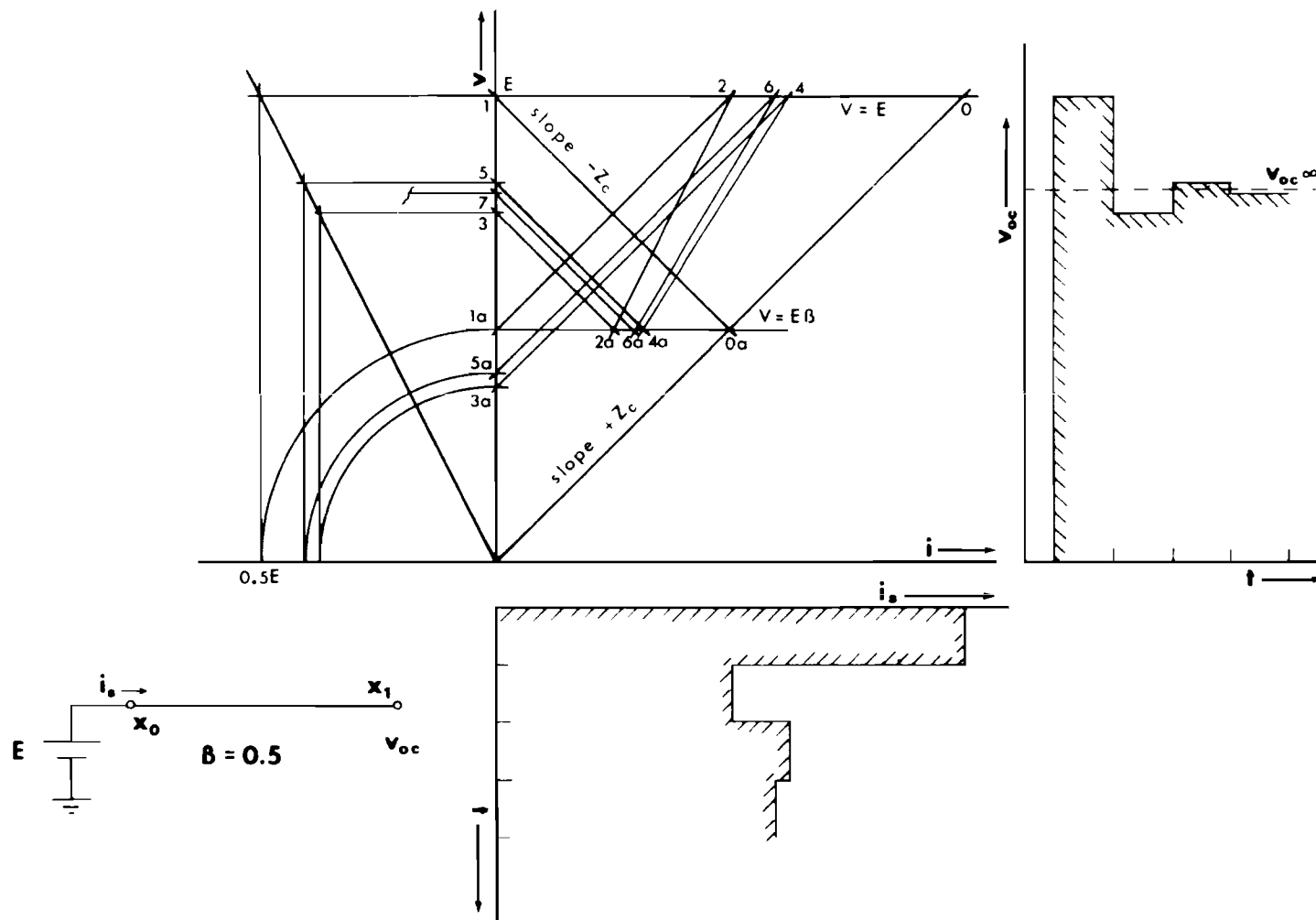


Fig. (9.6) Charging a line from a d.c. source.



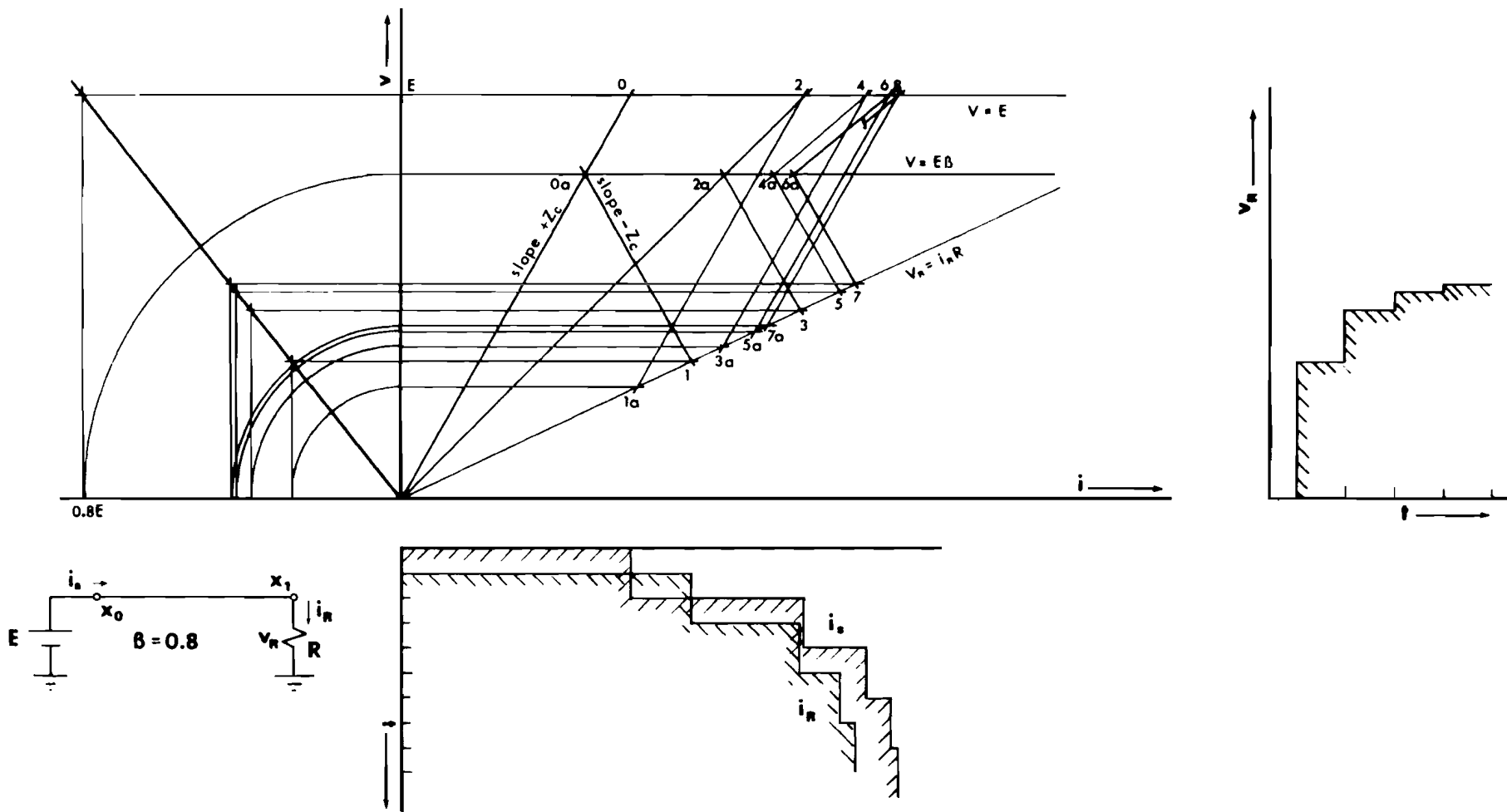


Fig. (9.8) Voltage step applied to a line terminated in a resistance  $R < Z_c$ .

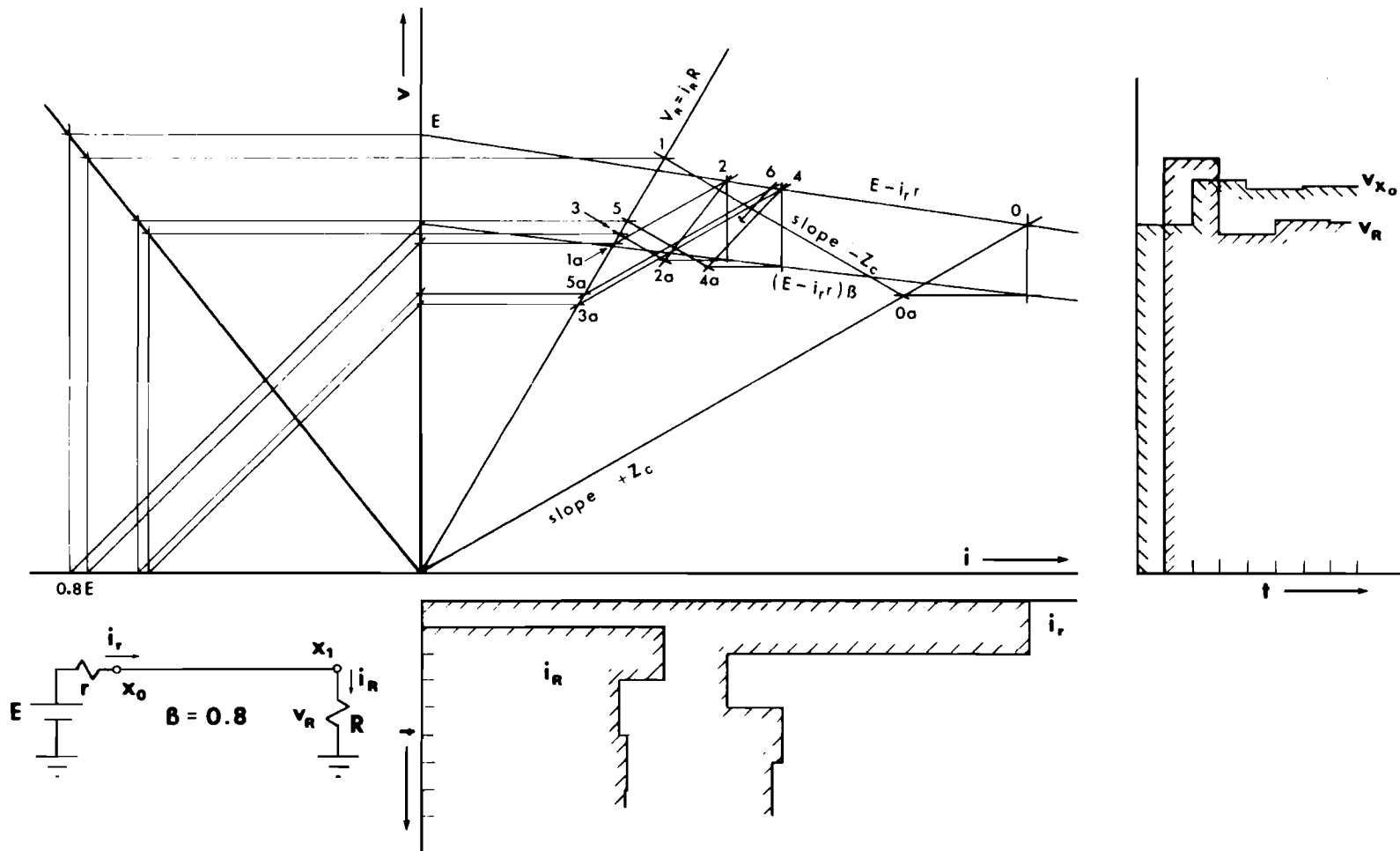


Fig. (9.9) Voltage step applied through a source resistance to a line terminated in a resistance  $R > Z_c$ .

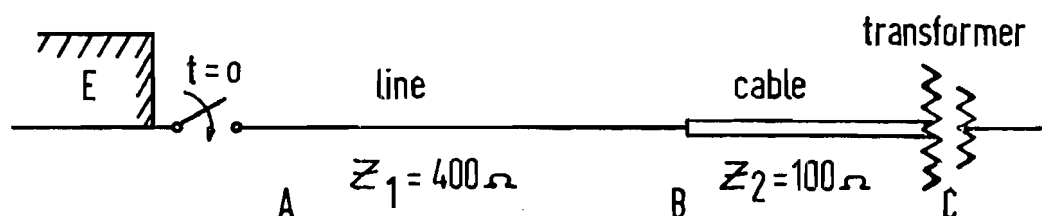
propagation is drawn to intersect  $v_R = i_R R$  at point '1'. Point '1a' will also lie on  $v_R = i_R R$  since this forms a straight line from point '1' to the origin. The procedure then follows in the same manner as the previous two examples. Alternatively the construction shown in Fig. (9.5) could have been used to find points '1a', '3a', '5a', ....

(b) Case for  $R > Z_C$  and a source internal resistance  $r$

The construction for this case is shown in Fig. (9.9). As in the prior example the equation at end  $x_1$ , is  $V_R = i_R R$ . At end  $x_0$  the equation is  $v = (E - i_r r)$ . The construction of Fig. (9.5) has been used to obtain  $(v_{x_0}^\beta, i_{x_0}^\beta)$ . The diagram has been constructed in the same manner as the previous three.

9.2.4) Application Example: Cable Protection of a Transformer from a Voltage Surge

In an example given by ARLETT (1966, Part 2) the protection of a transformer by a length of cable from a rectangular voltage surge is considered. In his solution ARLETT has assumed a lossless representation of both the transmission line and cable. Here we will repeat this example but will take into account surge attenuation in the cable. An attenuation factor for the length of cable of  $\beta = 0.9$  will be assumed. The solution for a lossless transmission line and cable has been carried out (Fig. (9.11)) and the results obtained for the lossless and lossy cables are compared in Figs (9.13) and (9.14).



Transit time of line and cable are equal

Fig. (9.10) Cable protection of a transformer from a voltage surge

(a) Graphical Solution

The problem notation used is that given in Fig. (9.10). The solution for the lossy cable is given in Fig. (9.12) and follows the techniques already outlined. In this solution the equations of propagation along the overhead transmission line are continuous straight lines with slopes  $\pm Z_1$ . The equations of propagation along the cable are dashed straight lines with slopes  $\pm Z_2$ . The impedance of the transformer,  $Z_3$ , gives a cable terminal equation at point  $C$  ( $V_C = Z_3 I_C$ ) which is represented by a straight line of slope  $+Z_3$  passing through the origin. The equation applying at point  $A$  on the transmission line is  $V = E$  where  $E$  in this case is 100 Kv.

The method used to obtain the solution at point  $B$  is more easily seen in the lossless solution, Fig. (9.11), remembering that for the lossless solution  $\beta = 1.0$ . The step voltage,  $E$ , is applied at line end  $A$ . Point  $OA$  gives the solution immediately the surge is applied. A forward propagating wave travels along the line to point  $B$ . At point  $B$  one transition time later the forward propagating wave is described by:

$$(v + Z_1 i)_{1B} = (v + Z_1 i)_{0A}$$

Also at point B at that time the equation of backward propagation along the cable, describing the cable initial conditions is:

$$(v - Z_2 i)_{1B} = (v - Z_2 i)_{0C} = 0$$

These two equations solved simultaneously give solution point 1B. From point B a wave propagates forward along the cable to cable end C, solution point 2C, and a wave propagates backward to line end A, solution point 2A. From these points the solution at line point B, three transit times after the voltage was applied is obtained by solving the equations:

$$(v + Z_1 i)_{3B} = (v + Z_1 i)_{2A}$$

$$(v - Z_2 i)_{3B} = (v - Z_2 i)_{2C}$$

giving the solution point 3B. The rest of the solution is continued in the same manner.

For the lossy cable solution the same procedure is used except that now the equations of propagation along the cable are modified to take attenuation into account by using the construction shown in Fig. (9.3).

#### (b) Comparison of Results

Comparing the two solutions it is seen that the equations of propagation for the lossy cable fall progressively below those for the lossless cable. This leads to significant differences in the results obtained from the two solutions. In Fig. (9.13) the voltages at points B and C are compared. It is seen that for the lossy cable the

maximum voltage appearing at the transformer is significantly smaller. Comparing the currents at points B and C (Fig. (9.14)) it is seen that the transformer current is again lower for the lossy cable. The current at B however is quite different. Had a cable with greater surge attenuation been considered, or the transmission line surge attenuation taken into account, an even greater difference between the two solutions would have resulted.

This example illustrates the way in which a solution using lossless transmission elements rapidly diverges from a solution using lossy transmission elements. The use of a lossless representation of lossy transmission systems is satisfactory for only the most cursory examination of a travelling wave problem.



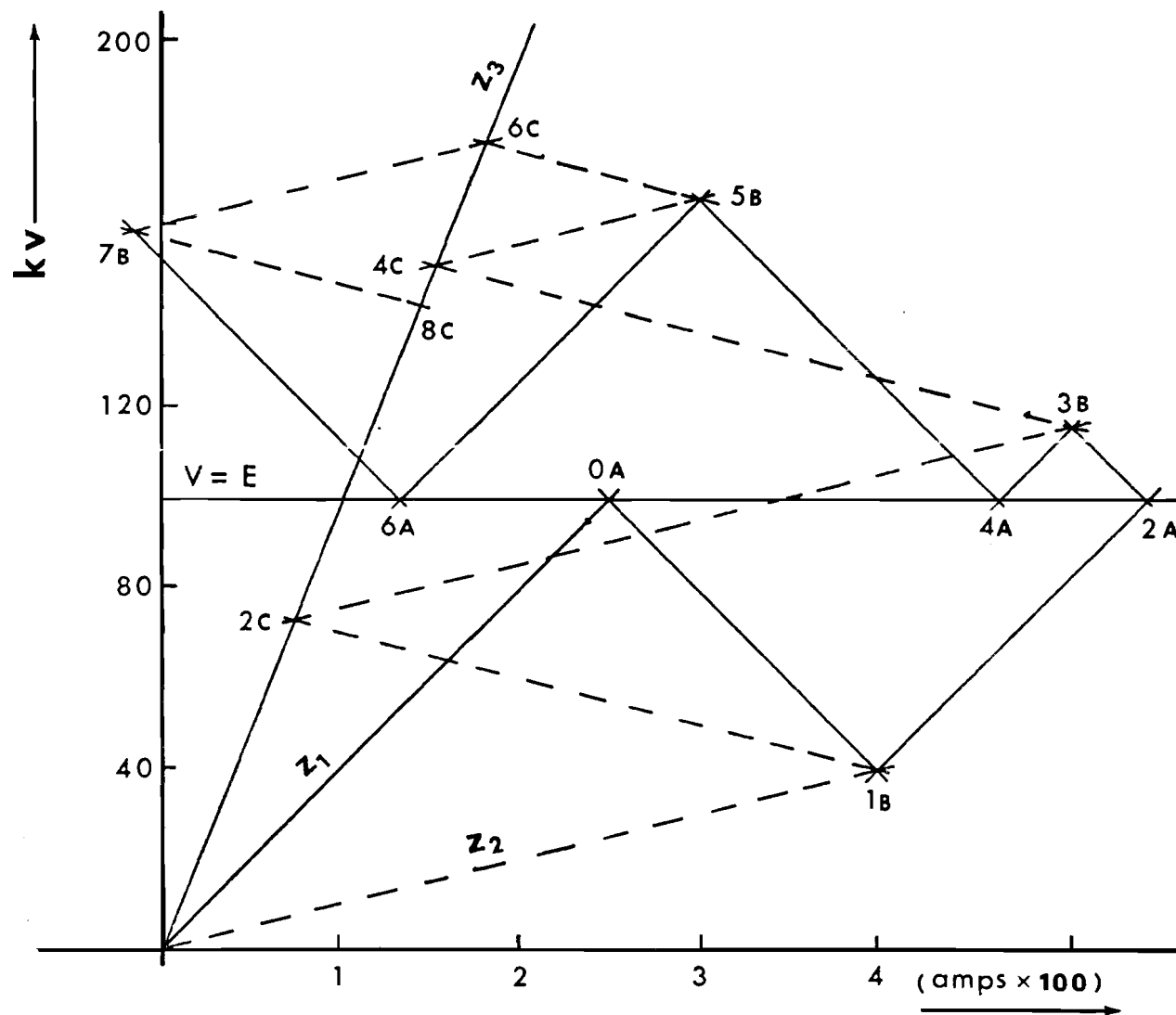
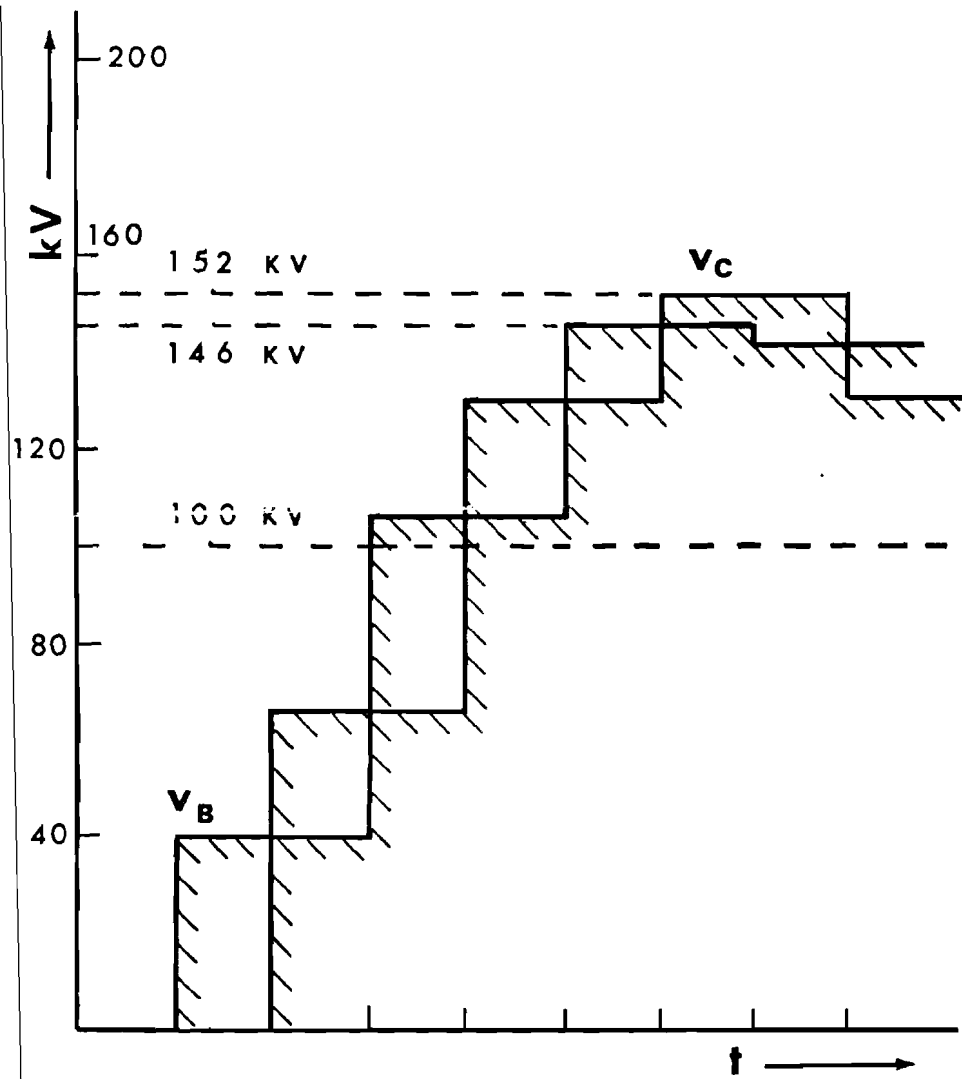
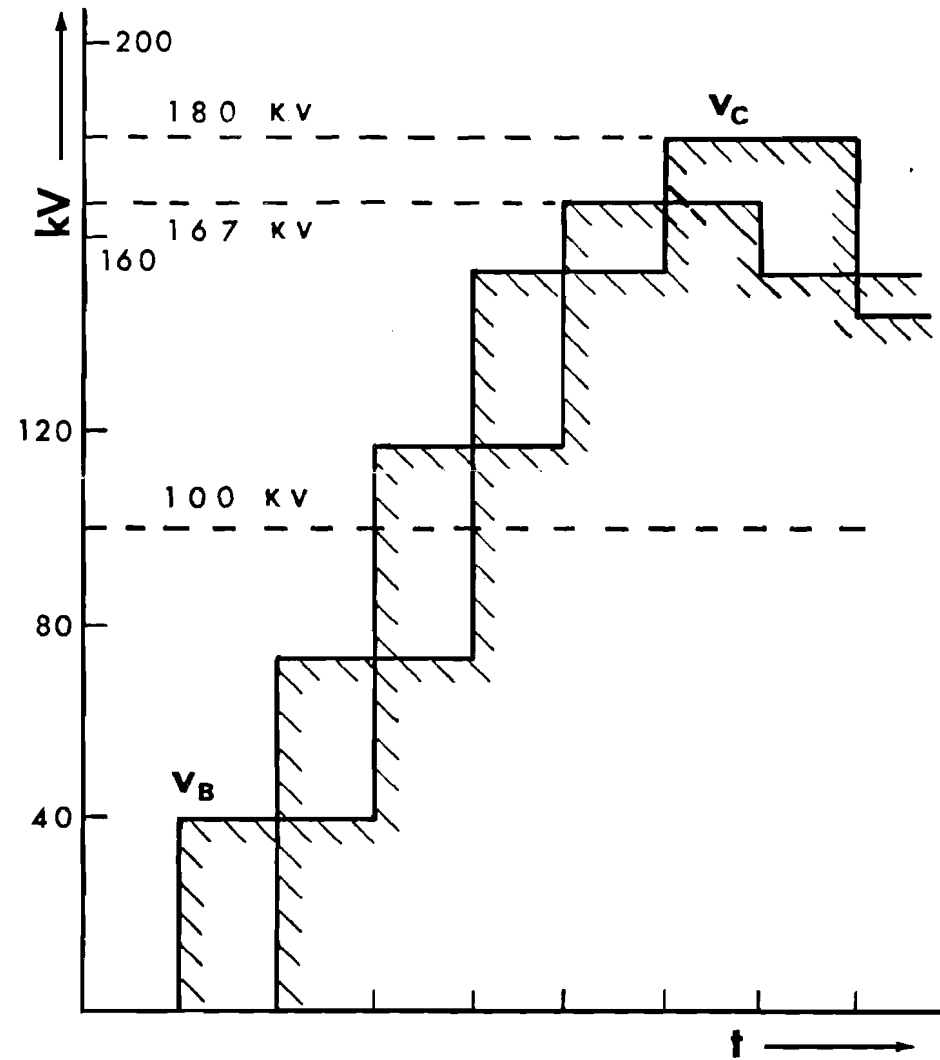


FIG [9.11] SOLUTION FOR LOSSLESS CABLE



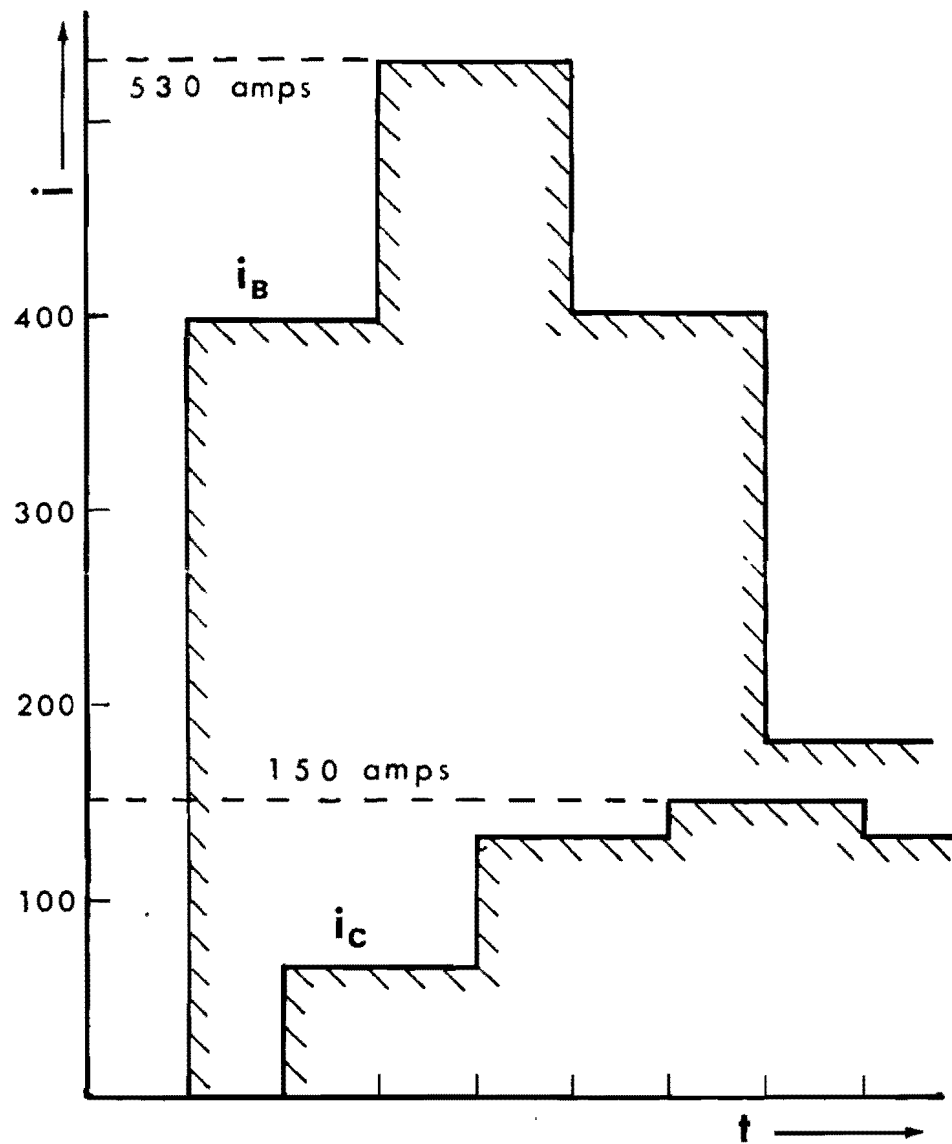


LOSSY CABLE

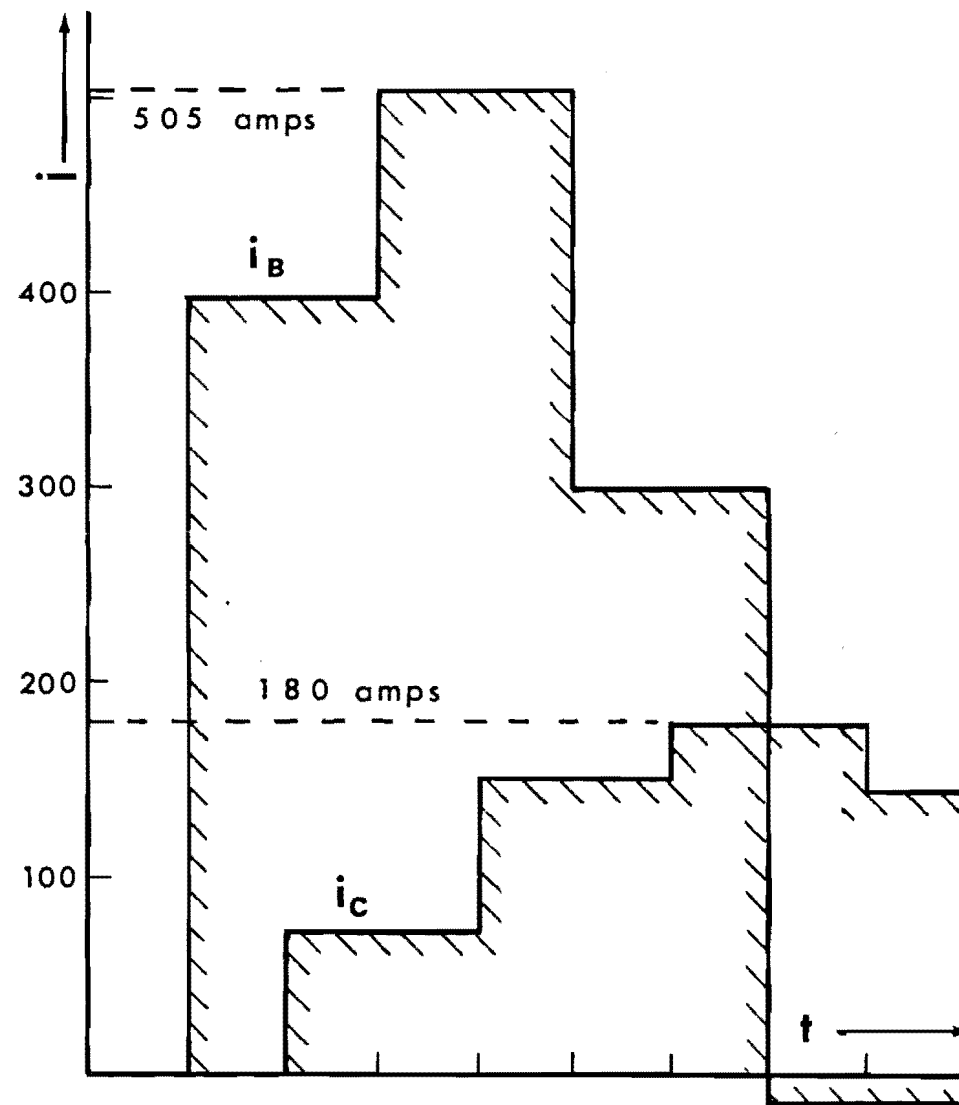


LOSSLESS CABLE

FIG (9.13) VOLTAGE AT POINTS B AND C



LOSSY CABLE



LOSSLESS CABLE

FIG (9.14) CURRENT AT POINTS B AND C

## SECTION 10

THE METHOD OF CHARACTERISTICS APPLIED TOTHE SERIES RESISTANCE LINE

It was pointed out in section 8.2.5) that although low loss and distortionless line approximations can be used to represent a series resistance line for short periods of time after a wavefront has passed, they have significant shortcomings when applied over long periods of time. Unfortunately it is often required in power systems analyses to have a solution which is valid for both short and long term transients. Thus it is necessary to seek ways of obtaining general solutions for the more complex series resistance line. Technically it is feasible to apply CARSON'S step voltage solution (section 4.6.1)) in conjunction with BEWLEY'S Lattice method (section 4.4.2)) in a superposition solution. However, this approach, as pointed out in section 4.6.2) is difficult to apply in the general situation and as pointed out in section 5) is not as flexible in its application as alternative time domain methods. What is required is a more simple, systematic approach to the solution of this problem in accordance with the criteria outlined in section 5). Such an approach will inevitably lead to an approximate solution but while being inexact, need not be inaccurate.

### 10.1) LUMPED SERIES RESISTANCE APPROXIMATION

This approximation has already been briefly discussed in section 4.6.4). It is a widely used way of approximating a series resistance transmission line and has been used in a comprehensive power systems electromagnetic transients program (DOMMEL, 1969). It will now be considered in detail and given a method of characteristics formulation. It will later be shown (section 10.2.1) part (b)) that this approximation turns out to be a special case of a more general approach to the solution of this problem. The propagation characteristics of this approximation are considered under this more general solution method (sections 10.2.2) part (d), 10.2.3), Figs (10.6) and (10.10)).

#### (a) Elementary line sections

A number of elementary line sections are possible, but to maintain symmetry of propagation and to enable the voltages and currents to be computed at unique points on the line the following line configuration is chosen. Each line section consists of two resistors, value  $\frac{R\Delta\ell}{2}$ , separated by a lossless transmission line of length  $\Delta\ell$ . This is shown in Fig. (10.1).

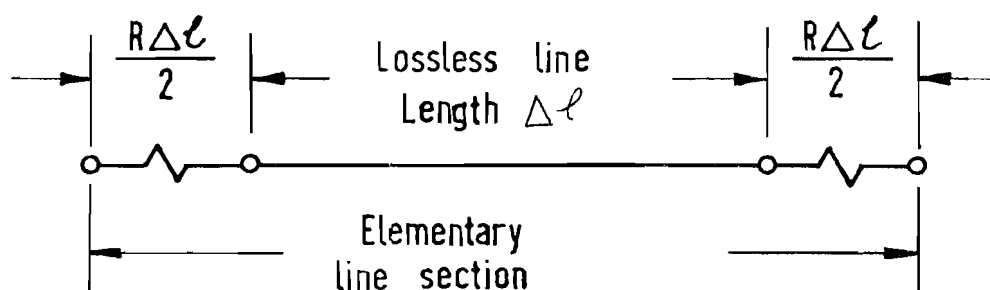


Fig. (10.1) Elementary Line Section

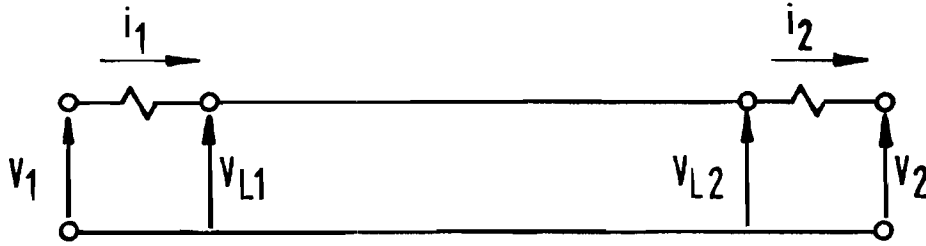


Fig. (10.2) Voltage and Current Convention

(b) Equations

The solution for the lossless line segment is given by:

$$v_{L2} + z_c i_2 = v_{L1} + z_c i_1 \quad \text{along } \frac{dt}{dx} = +\sqrt{LC}$$

$$v_{L1} - z_c i_1 = v_{L2} - z_c i_2 \quad \text{along } \frac{dt}{dx} = -\sqrt{LC}$$

At the line ends we have:

$$v_{L1} = v_1 - i_1 \frac{R\Delta\ell}{2}$$

$$v_{L2} = v_2 + i_2 \frac{R\Delta\ell}{2}$$

Substituting for  $v_{L1}$  and  $v_{L2}$  gives:

$$v_2 + (z_c + \frac{R\Delta\ell}{2})i_2 = v_1 + (z_c - \frac{R\Delta\ell}{2})i_1$$

$$\text{along } \frac{dt}{dx} = +\sqrt{LC}$$

(10.1)

$$v_1 - (z_c + \frac{R\Delta\ell}{2})i_1 = v_2 - (z_c - \frac{R\Delta\ell}{2})i_2$$

$$\text{along } \frac{dt}{dx} = -\sqrt{LC}$$

Equations (10.1) describe forward and backward propagation through each line section. For two cascaded sections with ends located at points  $(k-1)\Delta\ell$ ,  $k\Delta\ell$ ,  $(k+1)\Delta\ell$  along the  $x$  axis the solution at point  $(k\Delta\ell, t)$  is found from:

$$\begin{aligned}
\{v + (Z_C + \frac{R\Delta\ell}{2})i\}_{(k\Delta\ell, t)} &= \{v + (Z_C - \frac{R\Delta\ell}{2})i\}_{((k-1)\Delta\ell, t-\Delta t)} \\
\{v - (Z_C + \frac{R\Delta\ell}{2})i\}_{(k\Delta\ell, t)} &= \{v - (Z_C - \frac{R\Delta\ell}{2})i\}_{((k+1)\Delta\ell, t-\Delta t)}
\end{aligned}
\tag{10.2}$$

For convenience in computing, the solution would usually be obtained using cascaded pairs of line sections (section 10.2.1) part (d)).

## 10.2) SOLVING THE CHARACTERISTIC EQUATIONS OF A SERIES RESISTANCE LINE

The characteristic equations of a series resistance line are:

$$\begin{aligned}
(v + Z_C i)_{(x_s, t_s)} - (v + Z_C i)_{(x_1, t_1)} &= -R \int_{x_1}^{x_s} i \, dx \\
&\quad \text{along } C_1 \\
(v - Z_C i)_{(x_s, t_s)} - (v - Z_C i)_{(x_2, t_2)} &= -R \int_{x_2}^{x_s} i \, dx \\
&\quad \text{along } C_2
\end{aligned}
\tag{10.3}$$

(where the notation of Fig. (7.4), page 80 has been used).

In order to evaluate the line integrals, functional forms of  $i$  will be assumed. The resulting equations are no longer the characteristic equations but are approximations to them.

### 10.2.1) Use of Line Current Function Approximations

Three possible functions for  $i(x, t)$  along the characteristic curves will be examined.



(a) Constant Current Function Approximation

Assume  $i(x, t) = i(x_1, t_1)$  for  $x \geq x_1$  on the curve  $C_1$ ,

$$\text{i.e.} \quad \int_{x_1}^{x_s} i(x, t) dx = i(x_1, t_1) \Delta x_1$$

along  $C_1$ .

The integral along  $C_2$  may be similarly treated and equations (10.3) become:

$$\begin{aligned} (v + Z_c i)_{(x_s, t_s)} &= (v + Z_c (1 - \frac{R \Delta x_1}{Z_c}) i)_{(x_1, t_1)} \\ (v - Z_c i)_{(x_s, t_s)} &= (v - Z_c (1 - \frac{R \Delta x_2}{Z_c}) i)_{(x_2, t_2)} \end{aligned} \quad (10.4)$$

(b) Linear Current Function Approximation

Here  $i(x, t) = (i(x_s, t_s) - i(x_1, t_1))/\Delta x_1$ , for  $x \geq x_1$  on the curve  $C_1$ . This gives a trapezium rule approximation to the integrals.

$$\text{i.e.} \quad \int_{x_1}^{x_s} i(x, t) dx = \frac{\Delta x_1}{2} (i(x_1, t_1) + i(x_s, t_s))$$

along  $C_1$ .

Treating the integral along  $C_2$  in the same way we obtain from equations (10.3):

$$\begin{aligned} (v + Z_c (1 + \frac{R \Delta x_1}{2 Z_c}) i)_{(x_s, t_s)} &= (v + Z_c (1 - \frac{R \Delta x_1}{2 Z_c}) i)_{(x_1, t_1)} \\ (v - Z_c (1 + \frac{R \Delta x_2}{2 Z_c}) i)_{(x_s, t_s)} &= (v - Z_c (1 - \frac{R \Delta x_2}{2 Z_c}) i)_{(x_2, t_2)} \end{aligned} \quad (10.5)$$

It will be immediately apparent that equations (10.5) are identical to equations (10.2). Hence considering a series resistance line to be a cascaded series of lumped resistances separated by segments of lossless transmission line is the same as assuming that the line current varies in a linear manner along the characteristic curves of a distributed loss line.

(c) Exponential Current Function Approximation

Assume that  $i(x,t) = i(x_1, t_1) e^{-\frac{R(x-x_1)}{2Z_c}}$  for  $x \geq x_1$  along the curve  $C_1$ .

$$\int_{x_1}^{x_s} i(x,t) dx = \frac{2Z_c}{R} (1 - e^{-\frac{R\Delta x_1}{2Z_c}}) i(x_1, t_1)$$

along  $C_1$

Substituting for the integrals in equations (10.3) we obtain:

$$\begin{aligned} (v + Z_c i)_{(x_s, t_s)} &= (v - Z_c (1 - 2e^{-\frac{R\Delta x_1}{2Z_c}}) i)_{(x_1, t_1)} \\ (v - Z_c i)_{(x_s, t_s)} &= (v + Z_c (1 - 2e^{-\frac{R\Delta x_2}{2Z_c}}) i)_{(x_2, t_2)} \end{aligned} \quad (10.6)$$

(d) Summary of Results and Solution Grid

These equations all have the same form and their solutions for the case when  $\Delta x_1 = \Delta x_2 = \Delta x$  can be summarized as:

$$v(x_s, t_s) = (v(x_1, t_1) + v(x_2, t_2) + A(i(x_1, t_1) - i(x_2, t_2))) / 2$$

$$i(x_s, t_s) = (v(x_1, t_1) - v(x_2, t_2) + A(i(x_1, t_1) + i(x_2, t_2))) / B$$

(10.7)

where for approximation

$$(a) \quad A = z_c \left(1 - \frac{R\Delta x}{z_c}\right), \quad B = 2z_c$$

$$(b) \quad A = z_c \left(1 - \frac{R\Delta x}{2z_c}\right), \quad B = 2z_c \left(1 + \frac{R\Delta x}{2z_c}\right)$$

$$(c) \quad A = z_c \left(2e^{-\frac{R\Delta x}{2z_c}} - 1\right), \quad B = 2z_c$$

Equations (10.7) can be used systematically in the  $(x,t)$  plane with a set of grid points corresponding to the characteristic curves as illustrated in Fig. (10.3).

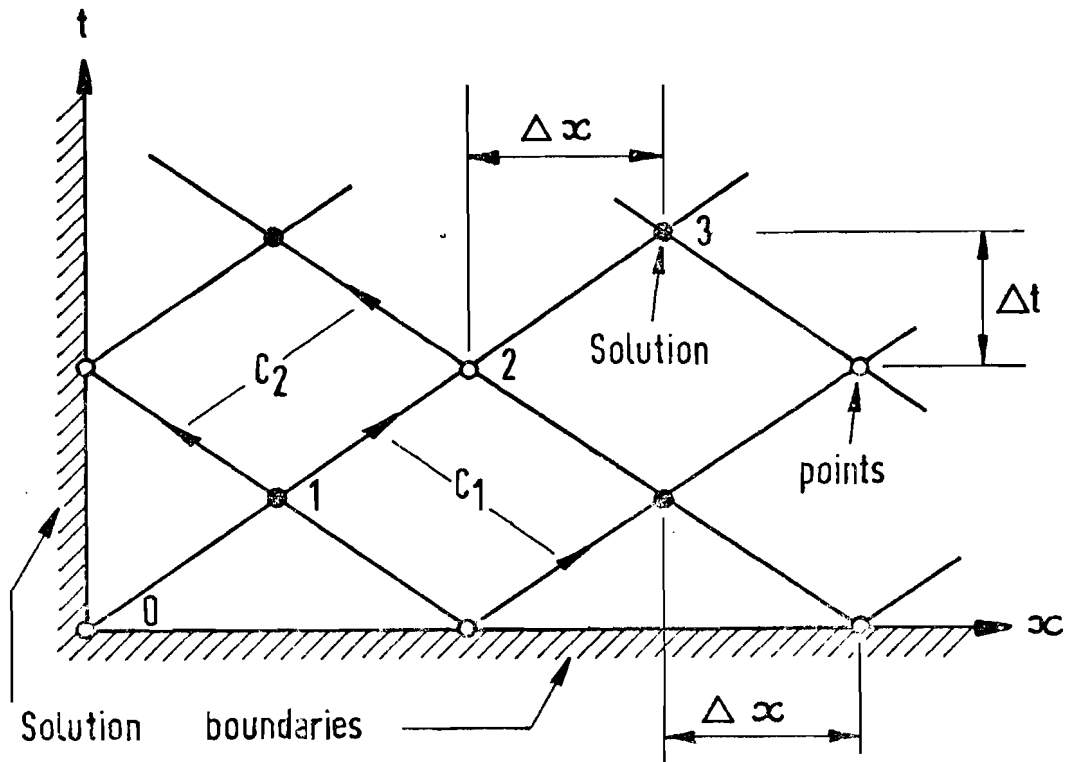


Fig. (10.3) A solution grid in the  $(x,t)$  plane corresponding to the characteristic curves of equations (10.3).

The amount of computing required can be reduced by approximately one half if the intermediate solution points, which do not correspond to a rectangular grid, are eliminated. These points are shown as black in Fig. (10.3). In terms of

the resulting rectangular grid the equations (10.6) become:

$$\begin{aligned} \left(v + \frac{B}{2}i\right)_{(x_s, t_s)} &= H(v + Ai)_{(x_s - 2\Delta x, t_s - 2\Delta t)} + k(v - Ai)_{(x_s, t_s - 2\Delta t)} \\ \left(v - \frac{B}{2}i\right)_{(x_s, t_s)} &= k(v + Ai)_{(x_s, t_s - 2\Delta t)} + H(v - Ai)_{(x_s + 2\Delta x, t_s - 2\Delta t)} \end{aligned} \quad (10.8)$$

where  $H = \left(\frac{1}{2} + \frac{A}{B}\right)$

$$k = \left(\frac{1}{2} - \frac{A}{B}\right).$$

### 10.2.2) Solution Properties

#### (a) Constant Current Function Approximation

Consider the transmission line to be initially at rest and apply a unit step of voltage at  $(x, t) = (0, 0)$ . Along the toe of the wave  $v(x_2, t_2)$  and  $i(x_2, t_2)$  (of equations (10.7)) are both zero.

Let  $v_k$  be the solution for  $v$  at the  $k^{\text{th}}$  solution point on the characteristic curve  $C_1$  that extends back through  $(x, t) = (0, 0)$  (Fig. (10.3)). Equations (10.7) then become:

$$v_k = \frac{1}{2} \left( v_{k-1} + i_{k-1} Z_c \left( 1 - \frac{R\Delta x}{Z_c} \right) \right)$$

$$i_k = v_k / Z_c$$

Thus for a unit step of voltage applied at the origin:

$$\begin{aligned} v_n &= \left( 1 - \frac{R\Delta x}{2Z_c} \right)^n \\ &= 1 - n(p\Delta x) + \frac{n(n-1)}{2!} (p\Delta x)^2 - \frac{n(n-1)(n-2)}{3!} (p\Delta x)^3 + \dots \end{aligned} \quad (10.9)$$

where  $p = \frac{R}{2Z_c}$

The exact solution is:

$$\begin{aligned}
 v_{n_{\text{exact}}} &= e^{-np\Delta x} \\
 &= 1 - n(p\Delta x) + \frac{n^2}{2!}(p\Delta x)^2 - \frac{n^3}{3!}(p\Delta x)^3 + \dots
 \end{aligned}
 \tag{10.10}$$

Hence the approximate solution will always be low at the toe of the wave. It is seen that as  $n$  increases and  $\Delta x$  decreases  $v_n$  tends to  $v_{n_{\text{exact}}}$ . It is also seen that the constant current approximation maintains the correct voltage current relation at the toe of the wave.

(b) Relation between Constant Current and Exponential Current Function Approximation

For the constant current approximation  $v_n < v_{n_{\text{exact}}}$ . We would like these to be equal and postulate that they can be if a suitable equivalent attenuation line resistance,  $R_{\text{eq}}$ , is used in place of the actual line resistance  $R$ .  $R_{\text{eq}}$  will be less than  $R$  and will be such that:

$$\left(1 - \frac{R_{\text{eq}}\Delta x}{2Z_c}\right)^n = e^{-\frac{R}{2Z_c}n\Delta x}$$

Taking the  $n^{\text{th}}$  root of both sides and rearranging gives:

$$R_{\text{eq}} = \frac{2Z_c}{\Delta x} \left(1 - e^{-\frac{R\Delta x}{2Z_c}}\right)
 \tag{10.11}$$

Replacing  $R$  by  $R_{\text{eq}}$  in equations (10.7) gives:

$$A = Z_c \left(2e^{-\frac{R\Delta x}{2Z_c}} - 1\right)$$

which is seen to be identical to the exponential current function approximation.

(c) Exponential Current Function Approximation

It is evident from the preceding equivalence that at the toe of the wave the solution for  $v$ , obtained from the exponential current approximation, will be exact and the correct voltage current relation will be maintained. Also, since this approximation is equivalent to a constant current series resistance line with a resistance lower than the true line resistance,  $i$  will tend to a value higher than the exact solution after the toe of the wave has passed. In the limit, as steady state conditions are reached  $i$  will be higher than the exact solution by an amount:

$$* \quad \Delta i = \frac{i \Delta R_L}{R_{eq_L} + R_T} \quad (10.12)$$

where  $i$  is the exact steady state solution

$R_{eq_L}$  is the total line equivalent attenuation resistance

$$\Delta R_L = R_L - R_{eq_L}$$

$R_L$  is the true total line resistance

for a line terminated in a resistance  $R_T$ . The effect is most severe for a line terminated in a short circuit. Equation (10.12) can be used to obtain an estimate of the final error in a solution.

For example: Consider a power system transmission line with  $Z = 250\Omega$ ,  $R = 0.01\Omega/1000$  yds, length 300 miles. It is desired to solve for the situation where a unit step of voltage is applied to the line when it is terminated in a short circuit. An exponential current function approximation is to be used in the characteristic equations and  $\Delta x$  for the

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\* Compensation Theorem

solution is chosen to be 30 miles (corresponding to 5 distance increments in the solution). Initially at and near to the toes of the resulting travelling waves the solution will be exact and near to exact. However, as time becomes large the line current will tend to become high. Finally, after a long time as steady state conditions are approached

$$\frac{\Delta i}{i} \rightarrow 0.15\%$$

from equation (10.12). Had the line not been terminated in a short circuit the error would have been less.

(d) Linear Current Function Approximation

Consider again applying a unit step of voltage to a line at rest and examining the solution on the characteristic curve  $C_1$  extending back through the origin (Fig. 10.3)). At the  $k^{\text{th}}$  solution point equations (10.7) become:

$$v_k = (v_{k-1} + (Z_c - \frac{R\Delta x}{2})i_{k-1})/2$$

$$i_k = v_k / (Z_c + \frac{R\Delta x}{2})$$

from which it is seen that the linear current approximation increases the transmission line surge impedance by  $\frac{R\Delta x}{2}$  above that of the actual line.

At the  $n^{\text{th}}$  solution point we obtain:

$$v_n = (1 + \frac{R\Delta x}{2Z_c})^{-n} \quad (10.13)$$

(See also equation (4.30) where the same result is obtained using a different approach.)

or

$$v_n = 1 - n(p\Delta x) + \frac{n(n+1)}{2!}(p\Delta x)^2 - \frac{n(n+1)(n+2)}{3!}(p\Delta x)^3 + \dots$$

Comparing this with the exact solution equation (10.10) it is seen that at the toe of a propagating wave this solution will always be high.

As before  $v_n$  can be made equal to  $v_{n_{\text{exact}}}$ , this time by increasing the line series resistance. This will however increase still further the surge impedance of the solution.

Again, letting  $R_{\text{eq}}$  be the equivalent attenuation line resistance and using the same procedure as in part (b) of this section we obtain for the linear current approximation:

$$R_{\text{eq}} = \frac{2Z_c}{\Delta x} \left( e^{\frac{R\Delta x}{2Z_c}} - 1 \right) \quad (10.14)$$

The new solution surge impedance becomes:

$$Z_{c_{\text{eq}}} = \left( Z_c + \frac{R_{\text{eq}}\Delta x}{2} \right) = Z_c e^{\frac{R\Delta x}{2Z_c}} \quad (10.15)$$

### 10.2.3) Computed Results

The step response of a series resistance transmission line has been computed for the constant current, linear current and exponential current function approximations. The results are compared with the exact solution of CARSON (1926). In order to make the results more generally applicable the per unit system used by HEDMAN (1971) is employed (see APPENDIX (13.1)). For convenience of comparison with both the voltage and current solutions, and to illustrate more readily the maintenance of correct voltage current relations at the toe of the wave, a surge impedance of 1 p.u. is assumed. Thus at the toe of the wave both voltage and current should have the same value. This will not affect the



trends that it is desired to show but will merely scale the curves according to the p.u. ~~section~~ system.

The results are shown in Figs (10.4)-(10.11). In these the exact solution is drawn as a continuous curve and the approximate solutions are drawn at the points where they are calculated. The solution step lengths used are:

$\Delta x = 0.1$	p.u.	i.e.	5	solution	increments	per	unit	length
$= 0.05$	p.u.	"	10	"	"	"	"	"
$= 0.025$	p.u.	"	20	"	"	"	"	"

All the trends and properties discussed in section 10.2.2) can be seen.

#### 10.2.4) Assessment of Solutions

The discrete lumped resistance approximation is frequently used to solve transient travelling wave problems in power systems. Although in the steady state this approximation yields an exact solution it is shown to have significant short comings in the transient situation. The wavefronts predicted will always be high for voltage and low for current. The solution surge impedance is greater than that of the actual line. If the approximation is modified by increasing the series resistance, the voltage attenuation at the wavefront can be made exact. Although this results in a good transient voltage approximation the transient current is even more in error than it was previously and the solution surge impedance is increased still further. Since there exist alternative solutions with identical form yet greater accuracy, use of this approximation in transient problems can no longer be justified.

Two alternative solutions in which the correct surge impedance is maintained have been derived from the method of characteristics solution of the T.L.E.s. By assuming the line current to propagate as a constant, a solution is obtained which is exact in the steady state but in which the wavefronts will always be low. By assuming the line current to propagate exponentially a solution is obtained in which the wavefronts are propagated exactly but in which the steady state current will be high. However, for this solution the steady state error can be estimated and is shown to be typically very low. This latter solution will thus yield results which have a high degree of accuracy to transient problems involving series resistance transmission lines. It requires calculation at only a small number of intermediate solution points, can accept arbitrary signals directly, involves only simple algebraic computations and can be formulated in a form which is ideal for systematic application on a digital computer.

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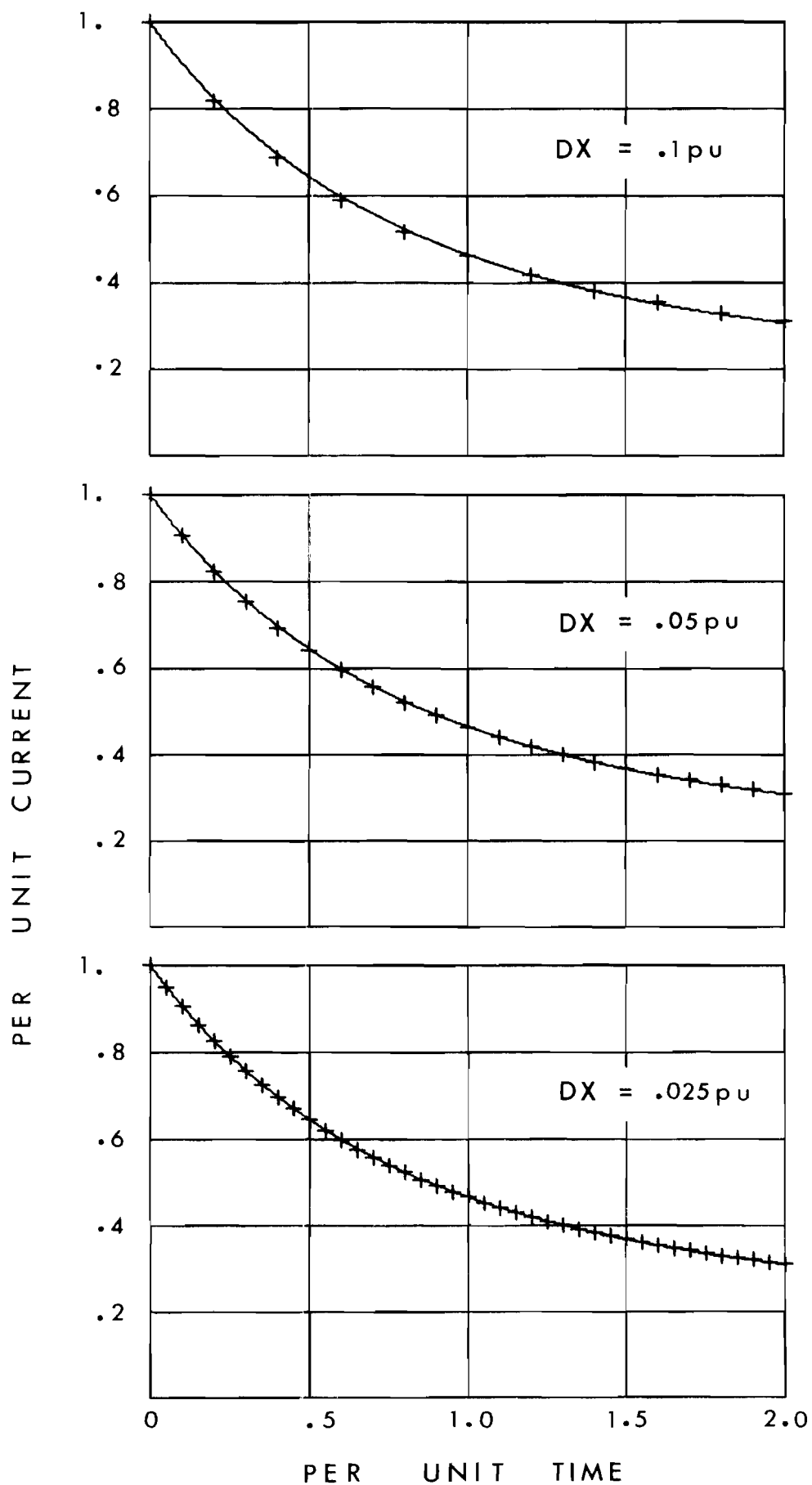


Fig. (10.4) Line input current. Constant current approximation.

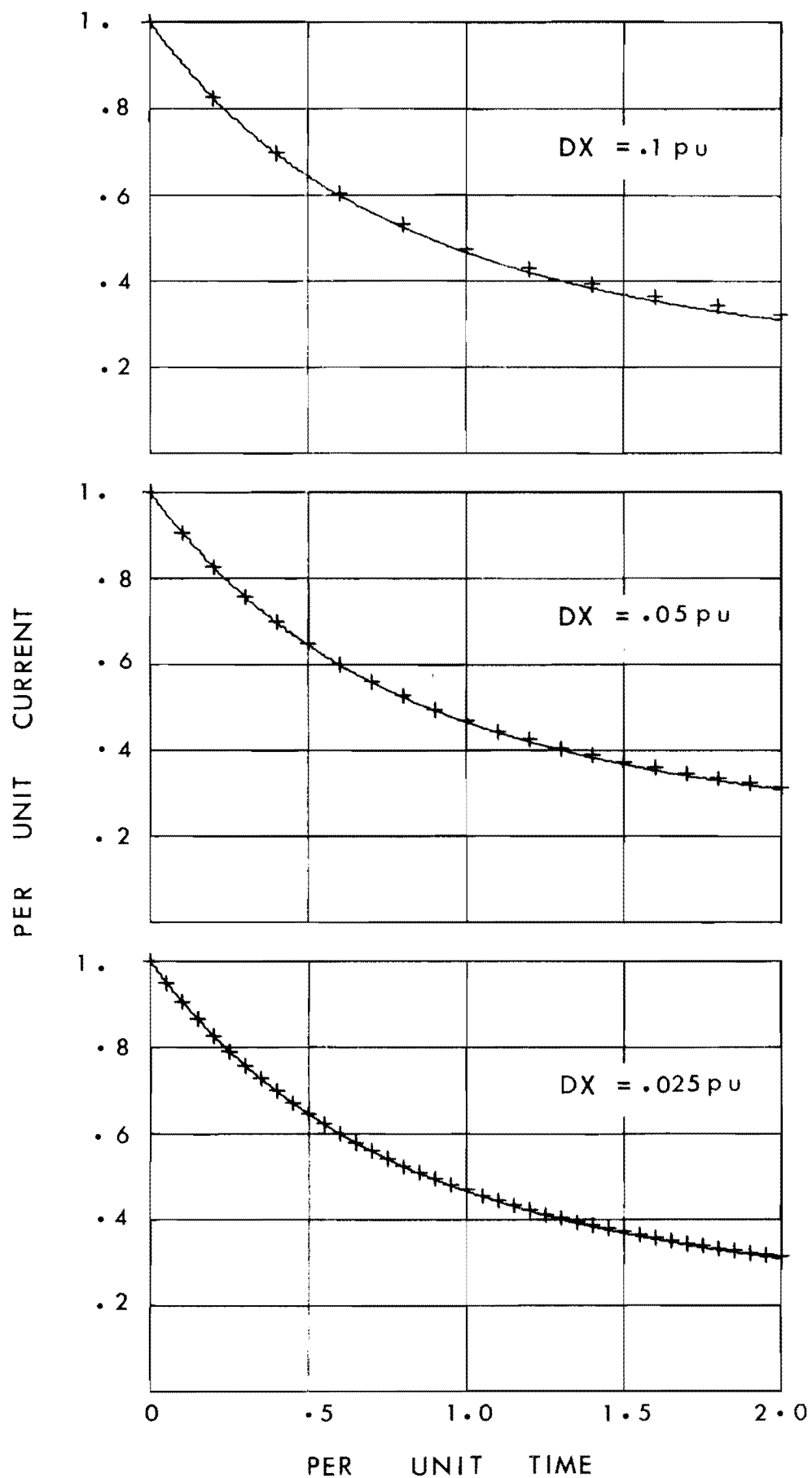


Fig. (10.5) Line input current. Exponential current approximation.

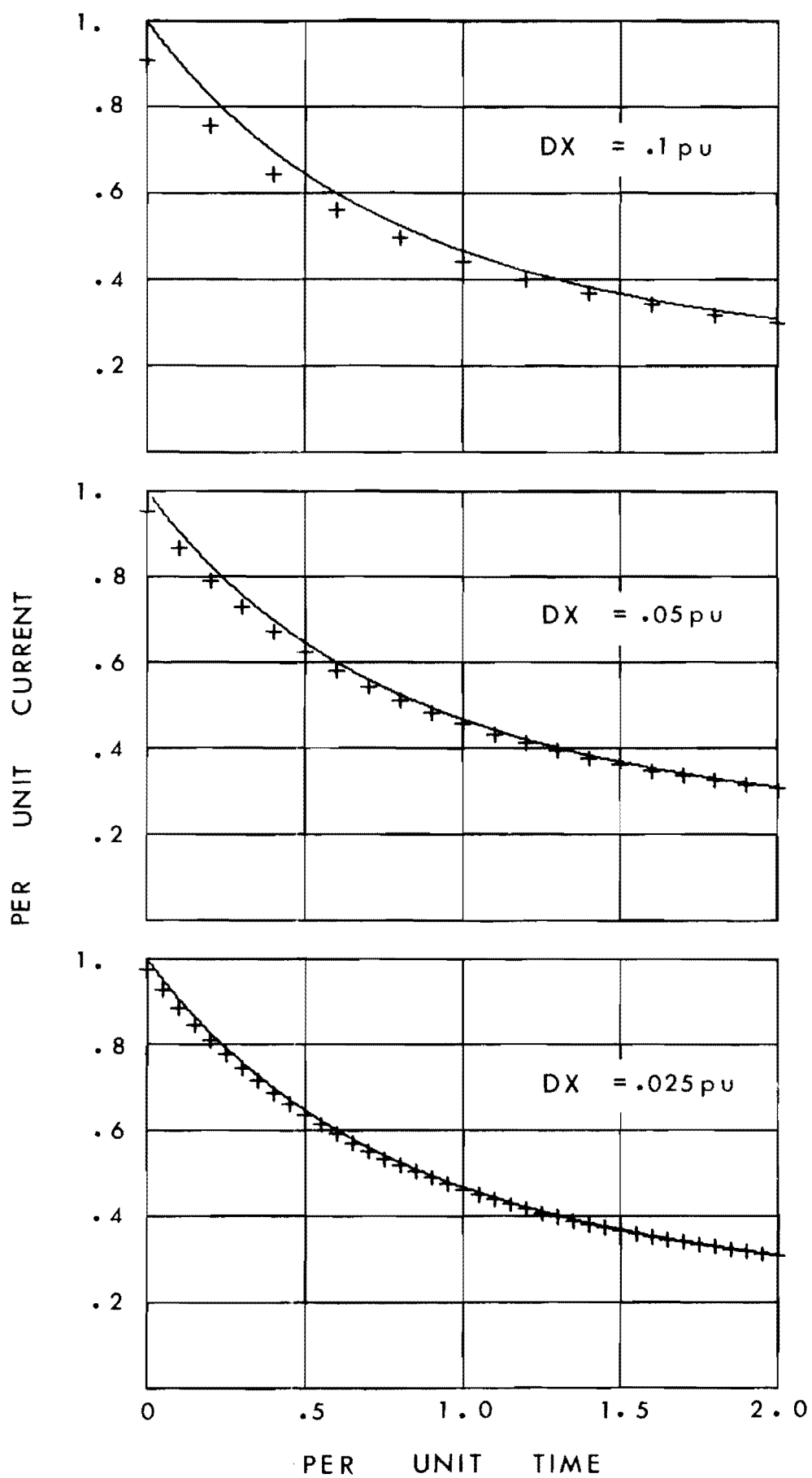


Fig. (10.6) Line input current. Lumped series resistance approximation.

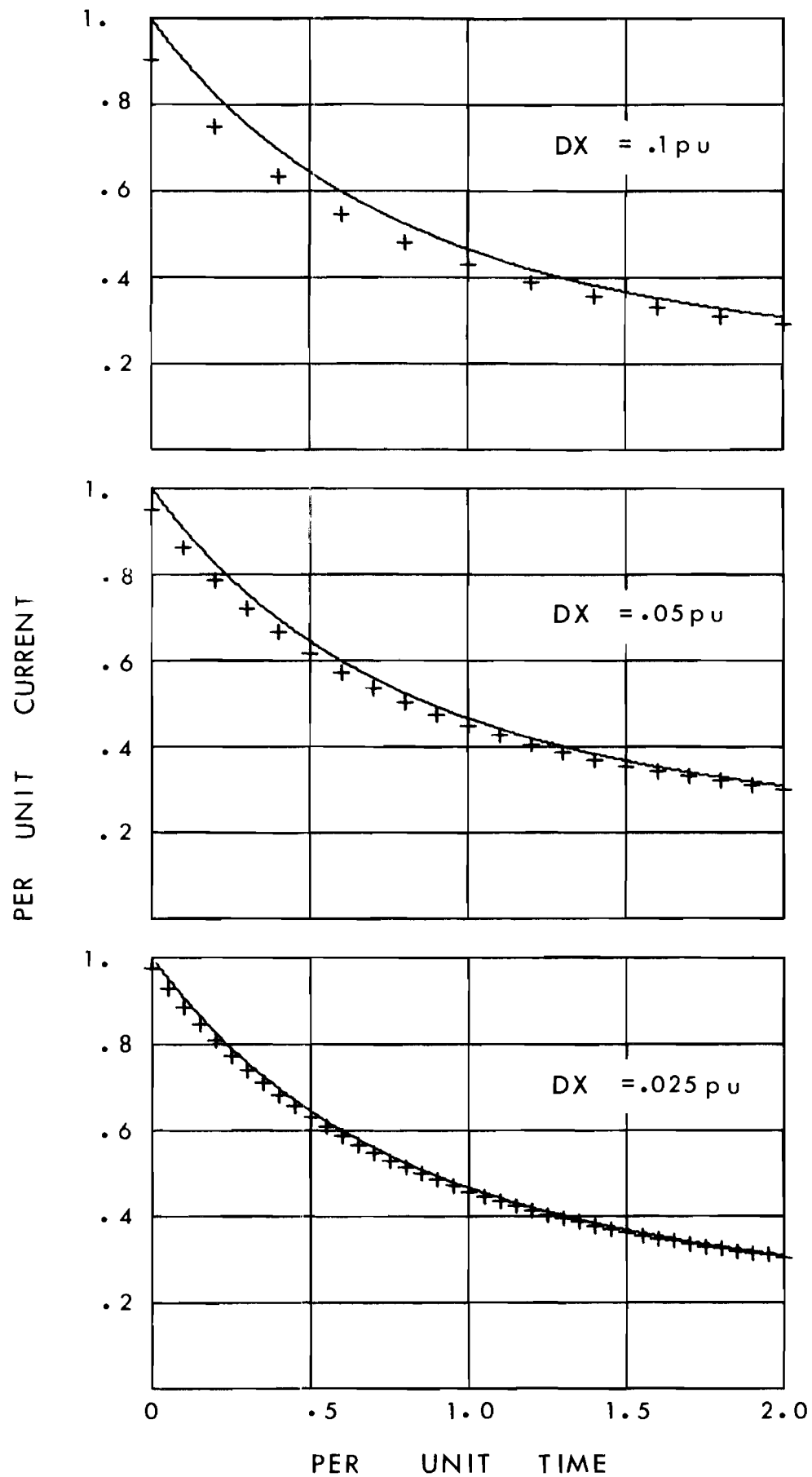


Fig. (10.7) Line input current. Modified lumped series resistance approximation.

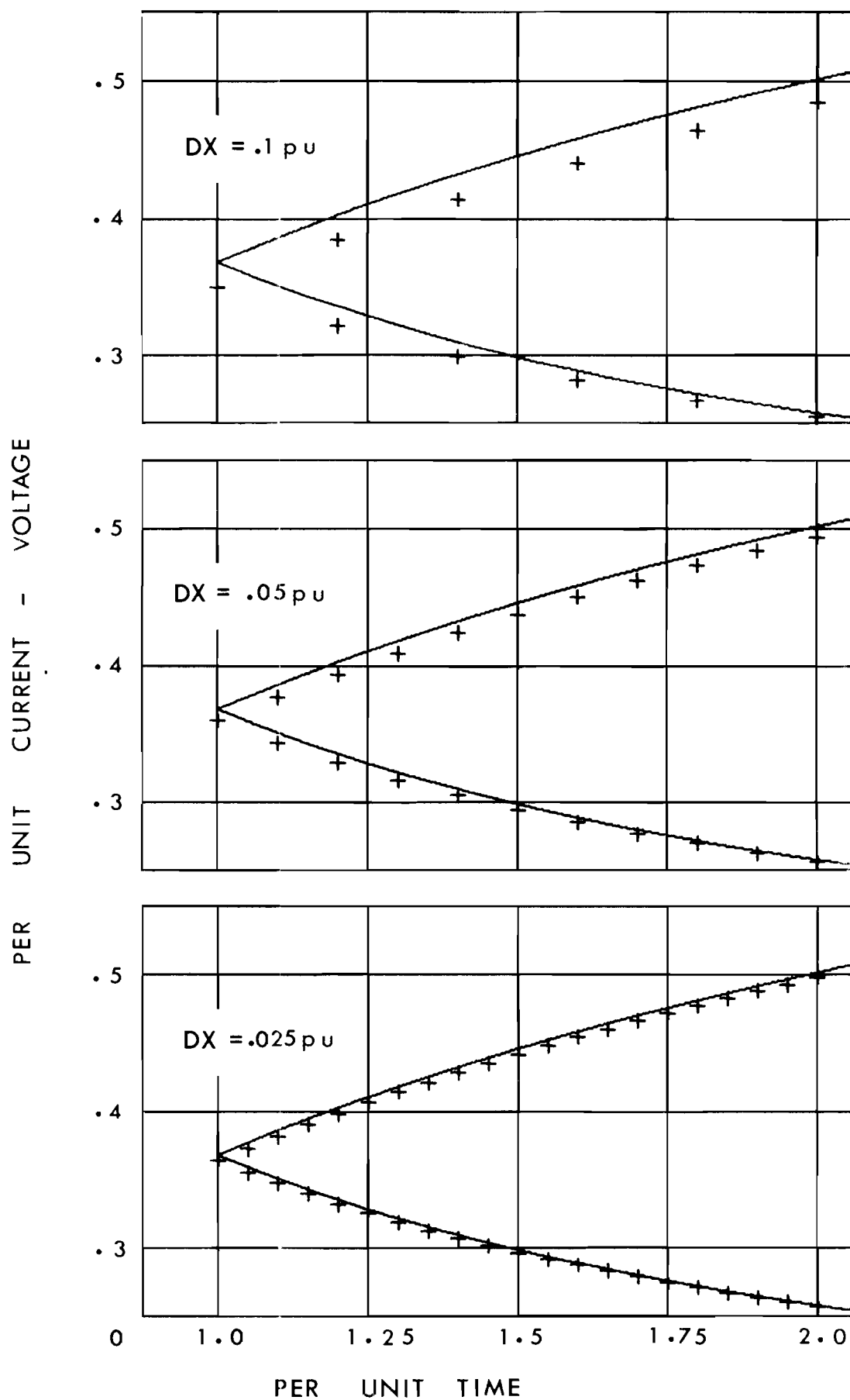


Fig. (10.8) Propagated wave at  $X = 1.0$  p.u. Constant current approximation.

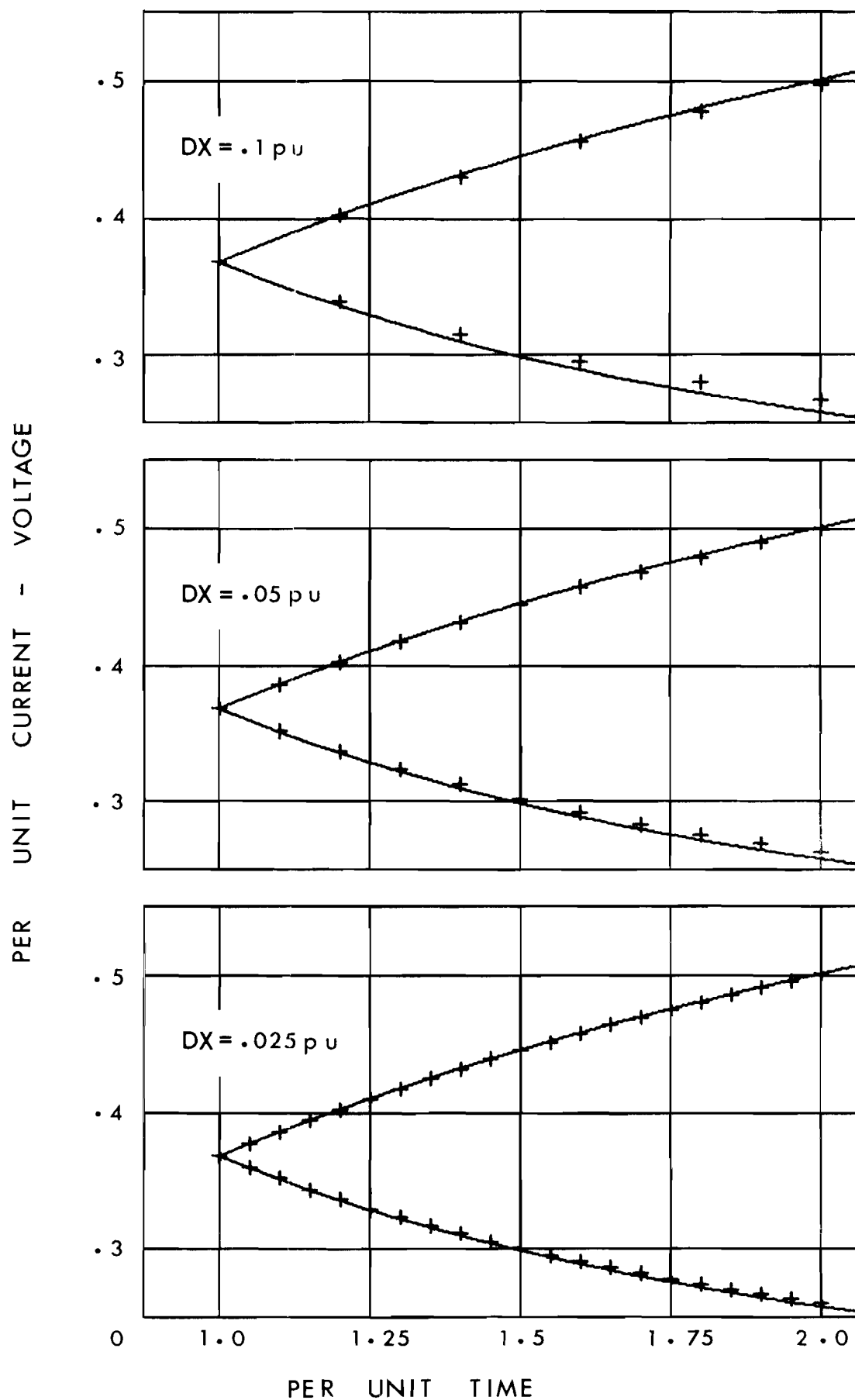


Fig. (10.9) Propagated wave at  $X = 1.0$  p.u. Exponential current approximation.



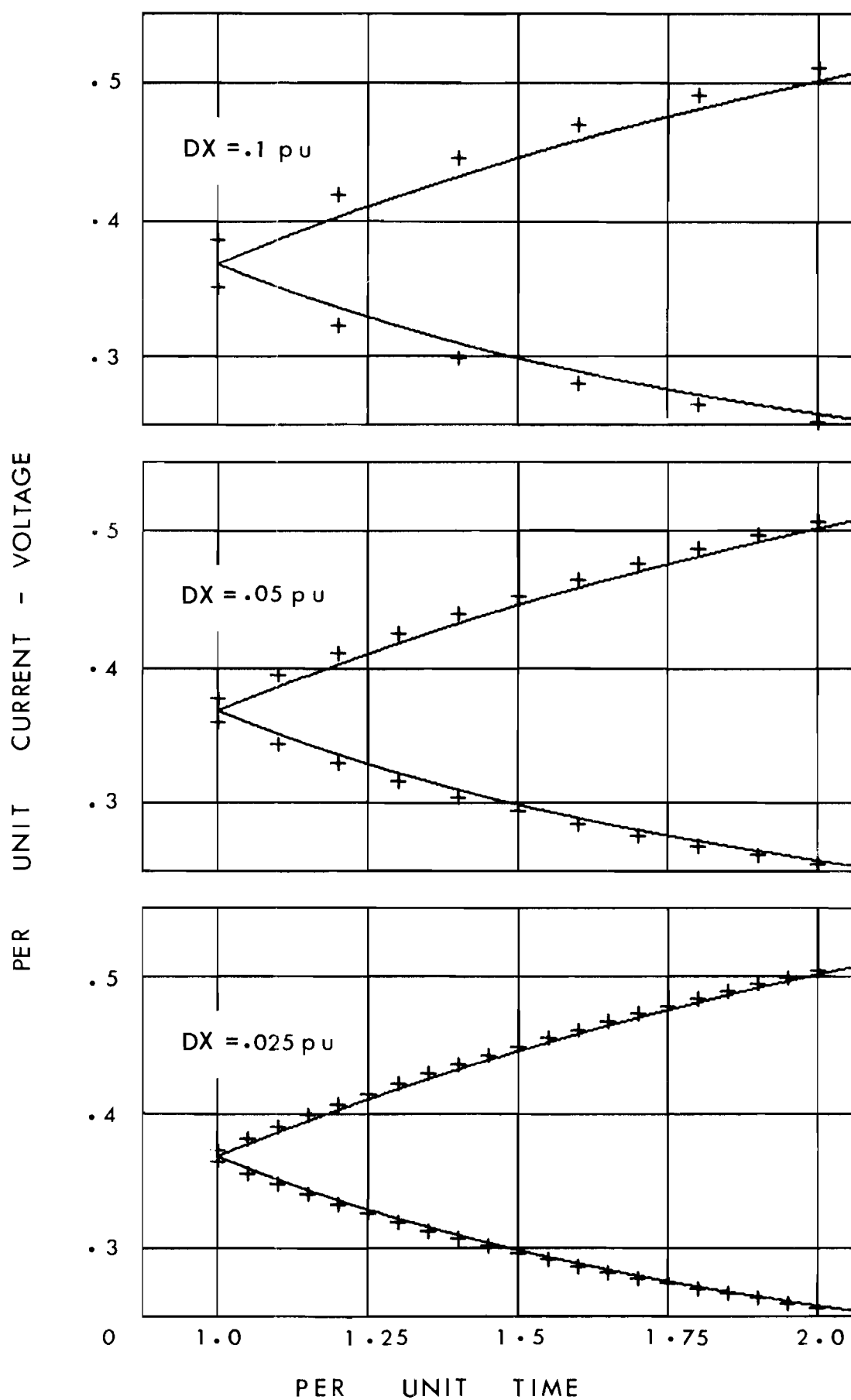


Fig. (10.10) Propagated wave at  $X = 1.0$  p.u. Lumped series resistance approximation.

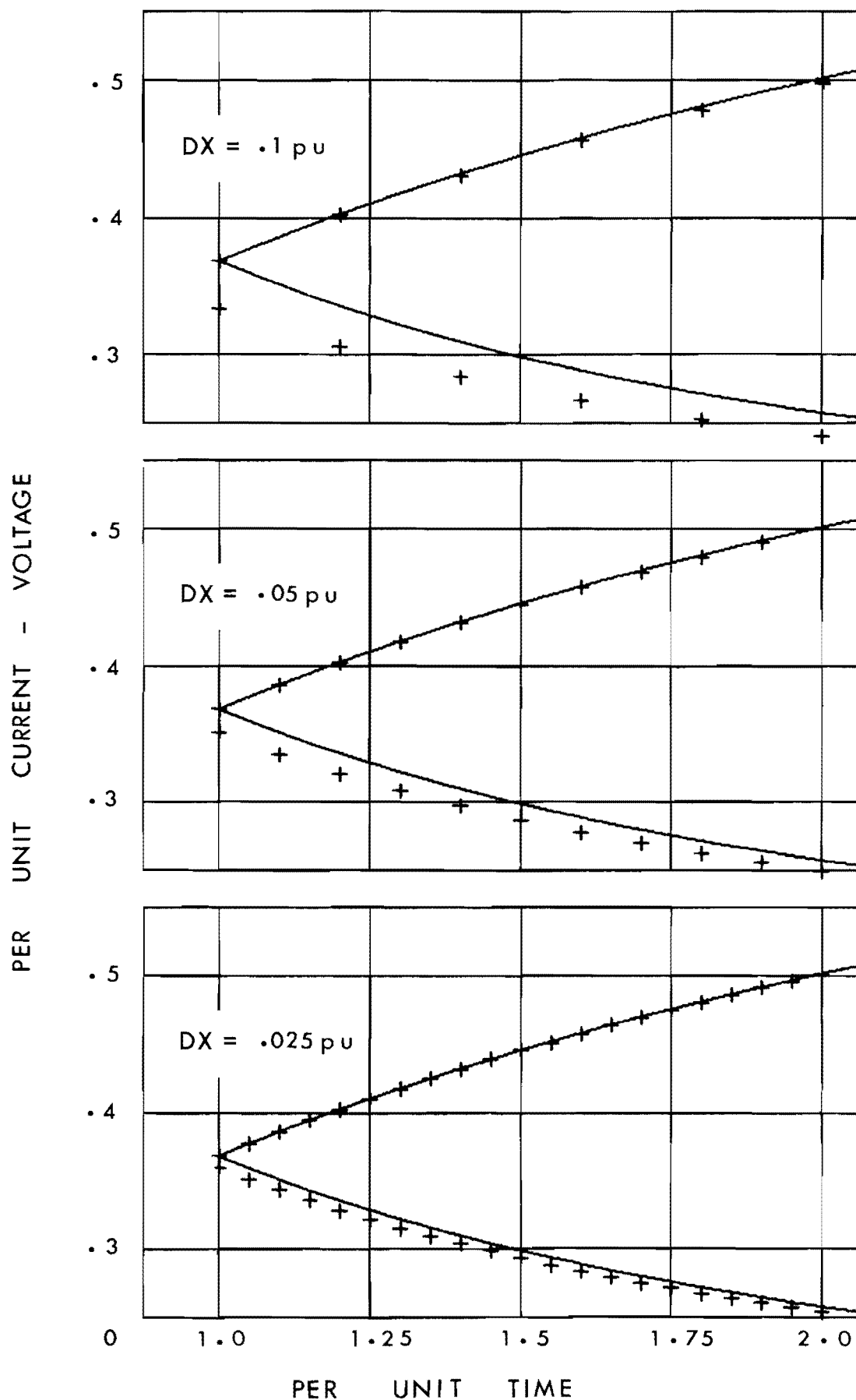


Fig. (10.11) Propagated wave at  $X = 1.0$  p.u. Modified lumped series resistance approximation.

## SECTION 11

MUTUALLY COUPLED MULTIPLE CIRCUIT TRANSMISSION LINES

In most transient situations met in power systems mutually coupled multiple circuit transmission lines are involved. Sometimes these can be reduced to an equivalent single circuit to which can be applied all the solution techniques previously established. Often, however, the phenomena being studied are highly unbalanced and such a reduction will lead to results that are inaccurate and can be quite misleading. It is necessary therefore to have ways of treating these transmission lines. In the time domain two approaches can be used. Direct application of the solution method or, where possible, applying matrix methods to reduce the problem to the solution of a number of single circuit uncoupled transmission lines. Both will be considered in relation to the method of characteristics.

Throughout this section the solution of a mutually coupled two circuit symmetrical transmission line will be worked through in full. This line was chosen because it is the simplest multiple circuit transmission line available to illustrate solution procedures and because it admits exact diagonalization into modal components thus enabling certain exact solutions to be produced. It also has the form of a two pole H.V.D.C. power transmission line to which the results obtained can be applied directly.

### 11.1) DIRECT APPLICATION OF THE METHOD OF CHARACTERISTICS

In section 4.4.8) it was stated that the method of characteristics could be applied directly to multiple circuit transmission lines using the same approach as for a single circuit transmission line. In section 6.2) the characteristic equations of a general  $n^{\text{th}}$  order transmission system were derived using a matrix formulation and in section 7.2) it was shown that for the T.L.E.s the characteristic equations can be derived using standard matrix operations. We are now concerned with solving the equations so derived.

#### 11.1.1) Characteristic Curves and Characteristic Equations

From section 7.1) it will be remembered that an  $n^{\text{th}}$  order transmission system has  $n$  characteristic curves in the  $(x,t)$  plane. Half of these apply in the forward direction and half in the reverse direction. For the linear transmission line these form sets of pairs of straight lines whose slopes have the same magnitude but opposite signs (e.g. Fig. (11.2)). The slopes of these lines are found as the eigenvalues of a matrix,  $[B]$ , derived from the T.L.E.s time derivative coefficient matrices (section 7.2)).

$$|[B] - \xi[I]| = 0 \quad (11.1)$$

where  $\xi$  are the eigenvalues of  $[B]$  and

$$[B] = \begin{bmatrix} 0 & [L] \\ [C] & 0 \end{bmatrix} \quad (\text{section 7.2))}$$

$\xi_j = \frac{dt}{dx} = \text{slope of characteristic curve } C_j \text{ in the } (x,t) \text{ plane.}$

To obtain the characteristic equations of the transmission line, the equations

$$[[B] - \xi_j[I]]^T \{\lambda\}_j^T = 0 \quad (11.2)$$

must be solved to give the eigenvector  $\{\lambda\}_j$ . Equations (11.2) are however, dependent, enabling only ratios of the elements of  $\{\lambda\}$  to be found. Write  $\{\lambda\} = \lambda_1 \{p\}$  where  $p_k = \lambda_k / \lambda_1$ . Now only the elements  $p_k$ ,  $k \neq 1$  need be found giving an independent set of equations. The characteristic equations are then (section 7.2)):

$$\lambda_1(\xi_j) \{p(\xi_j)\} \left\{ \frac{d}{dx} \{f\} + [h] \{f\} \right\} = 0 \quad (11.3)$$

where  $j = 1, 2, \dots, n$  and

$$\{f\} = \begin{Bmatrix} \{v\} \\ \{i\} \end{Bmatrix}, \quad [h] = \begin{bmatrix} 0 & [R] \\ [G] & 0 \end{bmatrix} \quad (\text{section 6.2))}$$

$\lambda_1(\xi_j)$  is an undetermined multiplier  $\neq 0$  which can be neglected.

Integrating equations (11.3) along their respective characteristic curves gives:

$$[\psi] \{f\} = \{w\} \quad (11.4)$$

where the  $j^{\text{th}}$  row of  $[\psi]$  is  $\{p(\xi_j)\}$  and the  $j^{\text{th}}$  element of  $\{w\}$  is:

$$w_j = \{p(\xi_j)\} \{f\}_{(x_j, t_j)} - \int_{x_j}^{x_s} \{p(\xi_j)\} [h] \{f\} dx \text{ along } C_j$$

which for a linear line becomes:

$$w_j = \{p(\xi_j)\} \left\{ \{f\} - |h| \int_{x_j}^{x_s} \{f\} dx \right\} \\ \text{along } C_j$$

The subscripts of equation (11.4) are in accordance with Fig. (11.1).

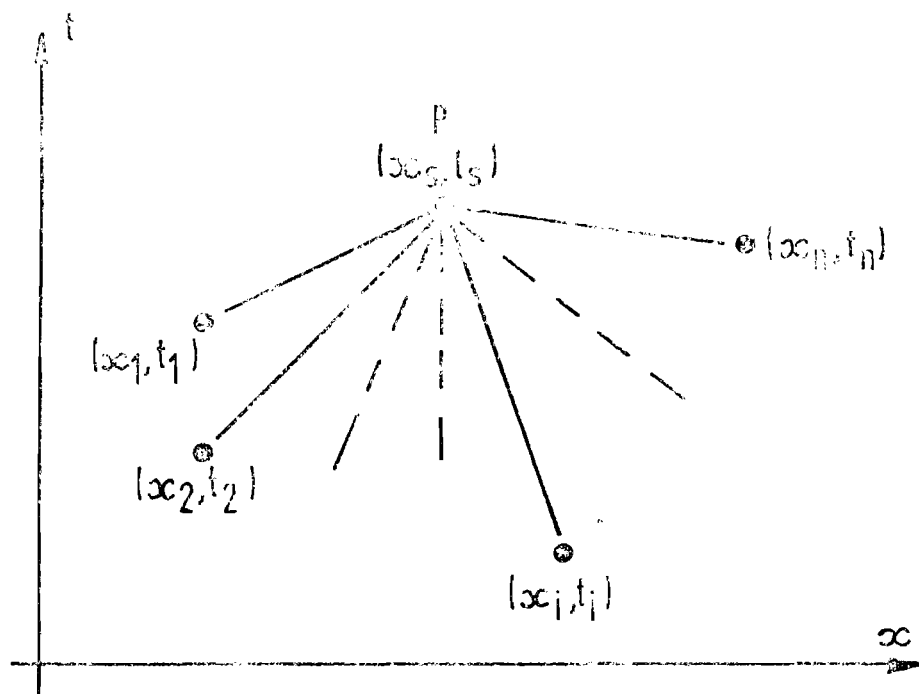


Fig. (11.1) Characteristic curves of an  $n^{\text{th}}$  order linear transmission system

#### 11.1.2) Application to a Mutually Coupled Two Circuit Symmetrical Transmission Line

Here equation (11.1) becomes:

$$\begin{vmatrix} -\xi & 0 & L_s & L_m \\ 0 & -\xi & L_m & L_s \\ C_s & C_m & -\xi & 0 \\ C_m & C_s & 0 & -\xi \end{vmatrix} = 0$$

This gives the polynomial:

$$0 = \xi^4 - 2(L_s C_s + L_m C_m) \xi^2 + (C_s^2 - C_m^2)(L_s^2 - L_m^2)$$

which has the solution:

$$\begin{aligned} \xi_1 &= +\sqrt{(L_s - L_m)(C_s - C_m)} \\ \xi_2 &= -\sqrt{(L_s - L_m)(C_s - C_m)} \\ \xi_3 &= +\sqrt{(L_s + L_m)(C_s + C_m)} \\ \xi_4 &= -\sqrt{(L_s + L_m)(C_s + C_m)} \end{aligned} \quad (11.5)$$

Equations (11.2) become:

$$\begin{bmatrix} -\xi_j & 0 & C_s & C_m \\ 0 & -\xi_j & C_m & C_s \\ L_s & L_m & -\xi_j & 0 \\ L_m & L_s & 0 & -\xi_j \end{bmatrix} \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{Bmatrix}_j = 0 \quad (11.6)$$

In terms of {p} equations (11.6) reduce to:

$$\begin{bmatrix} -\xi_j & C_m & C_s \\ L_m & -\xi_j & 0 \\ L_s & 0 & -\xi_j \end{bmatrix} \begin{Bmatrix} p_2 \\ p_3 \\ p_4 \end{Bmatrix}_j = \begin{Bmatrix} 0 \\ -L_s \\ -L_m \end{Bmatrix}$$

which when solved gives

$$\begin{aligned}
\{p(\xi_1)\} &= \{1, -1, z1, -z1\} \\
\{p(\xi_2)\} &= \{1, -1, -z1, z1\} \\
\{p(\xi_3)\} &= \{1, 1, z2, z2\} \\
\{p(\xi_4)\} &= \{1, 1, -z2, -z2\}
\end{aligned}
\tag{11.7}$$

$$\text{where } z1 = \sqrt{\frac{(L_s - L_m)}{(C_s - C_m)}} \quad \text{and} \quad z2 = \sqrt{\frac{(L_s + L_m)}{(C_s + C_m)}}$$

Thus for equations (11.4) we have:

$$[\psi] = \begin{bmatrix} 1 & -1 & z1 & -z1 \\ 1 & -1 & -z1 & z1 \\ 1 & 1 & z2 & z2 \\ 1 & 1 & -z2 & -z2 \end{bmatrix}$$

### 11.1.3 Conditions for an Exact Solution of a Two Circuit Symmetrical Transmission Line

The conditions required for analytical integration of the T.L.E.s characteristic equations are derived in section 7.3) and will be applied here to a symmetrical two circuit transmission line. The general loss matrix of such a line is:

$$[h] = \begin{bmatrix} 0 & & R_s & R_m \\ & 0 & R_m & R_s \\ G_s & G_m & & \\ G_m & G_s & & 0 \end{bmatrix} \tag{11.8}$$

(In a power systems context this loss matrix allows leakage across insulators onto non-perfectly conducting towers, resistive line conductors and an earth return resistance.)



The condition for exact integration of the characteristic equation along its particular characteristic curve is that along that curve:

$$\sum_{k=1}^4 \frac{\lambda_k h_{kj}}{\lambda_j} = \text{constant (invariant over } j)$$

where  $j = 1, \dots, 4$

Using identity (11.8) this can be expanded to:

$$\begin{aligned} \frac{\lambda_3}{\lambda_1} G_s + \frac{\lambda_4}{\lambda_1} G_m &= \frac{\lambda_3}{\lambda_2} G_m + \frac{\lambda_4}{\lambda_2} G_s = \frac{\lambda_1}{\lambda_3} R_s + \frac{\lambda_2}{\lambda_3} R_m \\ &= \frac{\lambda_1}{\lambda_4} R_m + \frac{\lambda_2}{\lambda_4} R_s \end{aligned} \quad (11.9)$$

Consider condition (11.9) along characteristic curve  $\frac{dt}{dx} = \xi_1$ :

$$\{\lambda\}_1 = \lambda_1 \{1, -1, z_1, -z_1\}$$

and substituting into (11.9) gives:

$$\left( \frac{R_s - R_m}{G_s - G_m} \right)^{-1} = \frac{1}{z_1^2} = \frac{C_s - C_m}{L_s - L_m} \quad (11.10)$$

$\{\lambda\}_2$  along  $\frac{dt}{dx} = \xi_2$  gives the same result.

For  $\frac{dt}{dx} = \xi_3$ ;  $\{\lambda\}_3 = \lambda_1 \{1, 1, z_2, z_2\}$  and condition (11.9) becomes:

$$\left( \frac{R_s + R_m}{G_s + G_m} \right)^{-1} = \frac{1}{z_2^2} = \frac{C_s + C_m}{L_s + L_m} \quad (11.11)$$

For a total exact solution  $R_s$ ,  $R_m$ ,  $G_s$  and  $G_m$  must be such that conditions (11.10) and (11.11) hold simultaneously. Referring to section 11.2.1) it will be seen that these conditions correspond to distortionless propagation of the component signals derived by transforming the T.L.E.s to

diagonal form. Such a solution cannot be said to be distortionless as the signal separates along diverging characteristic curves. However, along a given characteristic a signal is propagating with attenuation only.

#### 11.1.4) Function Approximations for Multiple Circuit Series Resistance Transmission Lines

As a consequence of the results established in section 10.2) only two function approximations will be considered here, constant and exponentially propagating line current. The approximations are the same as those used in section 10.2.1) but here will be formulated in terms of the general matrix elements of an  $n^{\text{th}}$  order transmission system's characteristic equations. The results will then be applied to a two circuit line.

Consider the  $j^{\text{th}}$  characteristic equation of an  $n^{\text{th}}$  order transmission system (equations (11.4)).

$$\begin{aligned} \{p(\xi_j)\}\{f\}_{(x_s, t_s)} \\ = \{p(\xi_j)\}\{f\}_{(x_j, t_j)} - \{p(\xi_j)\}[h] \int_{x_j}^{x_s} \{f\} dx \quad (11.12) \\ \text{along } C_j \end{aligned}$$

For a series resistance line equation (11.12) becomes:

$$\begin{aligned} \{p(\xi_j)\}\{f\}_{(x_s, t_s)} \\ = \{p(\xi_j)\}\{f\}_{(x_j, t_j)} - \sum_{k=1}^{n/2} R_k(\xi_j) \int_{x_j}^{x_s} i_k(x, t) dx \\ \text{along } C_j \quad (11.13) \end{aligned}$$

$$\text{where } R_k(\xi_j) = \sum_{\ell=1}^{\frac{n}{2}} p_{\ell}(\xi_j) \cdot h_{(\ell, k + \frac{n}{2})}$$

Separating  $\{f\}_{(x_j, t_j)}$  into  $\{v\}_{(x_j, t_j)}$  and  $\{i\}_{(x_j, t_j)}$  gives:

$$\begin{aligned} \{p(\xi_j)\}\{f\}_{(x_s, t_s)} &= \{P_1(\xi_j)\}\{v\}_{(x_j, t_j)} \\ &+ \{P_2(\xi_j)\}\{i\}_{(x_j, t_j)} - \sum_{k=1}^{\frac{n}{2}} R_k(\xi_j) \int_{x_j}^{x_s} i_k(x, t) dx \quad (11.14) \\ &\text{along } C_j \end{aligned}$$

where  $\{P_1\} = \{p_1, p_2, \dots, p_{\frac{n}{2}}\}$

$$\{P_2\} = \{p_{\frac{n}{2}+1}, p_{\frac{n}{2}+2}, \dots, p_n\}$$

For both constant current and exponential current function approximations the last two terms of equations (11.14) combine to a single term giving the equation:

$$\{p(\xi_j)\}\{f\}_{(x_s, t_s)} = \{P_1(\xi_j)\}\{v\}_{(x_j, t_j)} + \{A(\xi_j)\}\{i\}_{(x_j, t_j)} \quad (11.15)$$

(a) Constant current approximation

$$\text{Here } \int_{x_j}^{x_s} i_k(x, t) dx = i_k(x_j, t_j) \cdot \Delta x_j \text{ along } C_j$$

$$\text{where } \Delta_j = x_s - x_j$$

and

$$\{A(\xi_j)\}\{i\} = \sum_{k=1}^{\frac{n}{2}} \{p_{k+\frac{n}{2}}(\xi_j) \cdot i_k - \Delta x_j \cdot \sum_{\ell=1}^{\frac{n}{2}} p_{\ell}(\xi_j) \cdot h_{(\ell, k+\frac{n}{2})} \cdot i_k\}$$

Hence the vector  $\{A(\xi_j)\}$  is composed of elements:

$$A_k(\xi_j) = p_{k+\frac{n}{2}}(\xi_j) - \Delta x_j \cdot \sum_{\ell=1}^{\frac{n}{2}} p_{\ell}(\xi_j) \cdot h_{(\ell, k+\frac{n}{2})} \quad (11.16)$$

(b) Exponential current approximation

Here it is assumed that:

$$i_k(x, t) = i_k(x_j, t_j) e^{-\frac{R_k(\xi_j)}{2p_{k+\frac{n}{2}}(\xi_j)} (x - x_j)}$$

for  $t \geq t_j$  along curve  $C_j$ . Thus the integral:

$$\int_{x_j}^{x_s} i_k(x, t) dx = - \frac{2p_{k+\frac{n}{2}}(\xi_j)}{R_k(\xi_j)} (e^{-\frac{R_k(\xi_j)}{2p_{k+\frac{n}{2}}(\xi_j)} \Delta x_j} - 1)$$

along  $C_j$

and

$$\{A(\xi_j)\} \{i\} = \sum_{k=1}^{\frac{n}{2}} \{p_{k+\frac{n}{2}}(\xi_j) \cdot i_k - 2p_{k+\frac{n}{2}}(\xi_j) \cdot (e^{-\frac{R_k(\xi_j)}{2p_{k+\frac{n}{2}}(\xi_j)} \Delta x_j} - 1) \cdot i_k\}$$

which gives:

$$A_k(\xi_j) = p_{k+\frac{n}{2}}(\xi_j) \cdot (2e^{-\frac{R_k(\xi_j)}{2p_{k+\frac{n}{2}}(\xi_j)} \Delta x_j} - 1) \quad (11.17)$$

### 11.1.5) Errors Arising from Solution Grids

(a) Propagation time errors

It will be appreciated, on account of the differing slopes of the characteristic curves that it is difficult to match a simple rectangular solution grid to a composite grid of characteristics for an  $n^{\text{th}}$  order transmission system. This

is illustrated in Fig. (11.2).

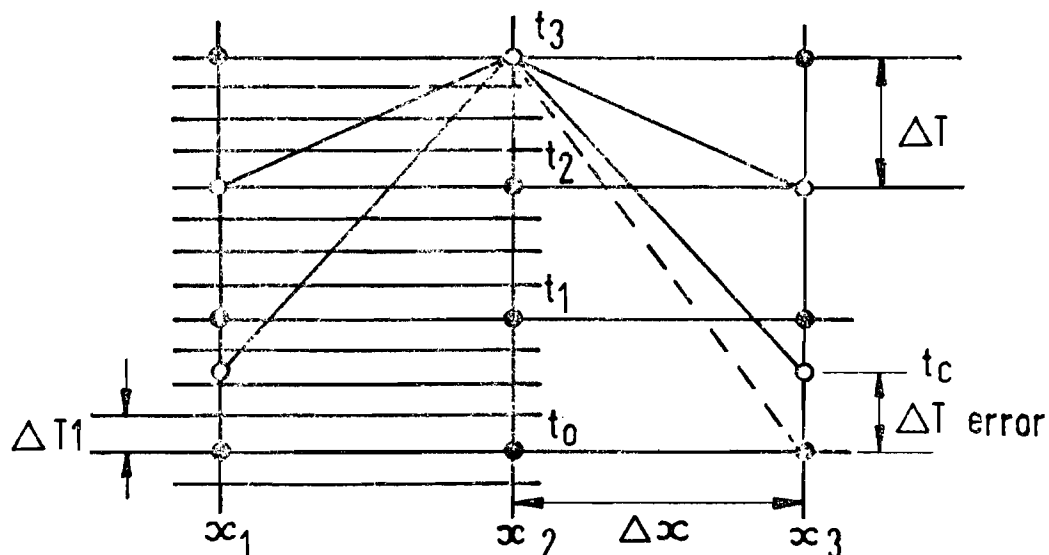


Fig. (11.2) Grid of characteristics and a rectangular solution grid for a linear two circuit transmission line

The variables are thus required to be known at points off the solution grid. A way of finding these would be to interpolate between  $t_0$  and  $t_1$  to obtain the signal at  $t_c$ . However, if signals of a discontinuous nature are being applied to or are propagating along the line, rapid accumulation of error will result. Interpolation is therefore not recommended for power system transient situations. Alternatively, and often more accurately, the signal at  $t_c$  can be assumed to be the same as at  $t_0$ . This, however, produces propagation time errors ( $\Delta T$  error in Fig. (11.2)) which may affect the superposition of waves to produce overvoltages on the line. This error is dependent on  $\Delta T$  and can be reduced by reducing the solution grid's time increment as shown in Fig. (11.2) with  $\Delta T_1$ . The effect of this on a solution can be seen in Figs (11.6) and (11.7).

(b) Incomplete solution along a characteristic curve

The characteristic curves of a two circuit line are shown in Fig. (11.3).

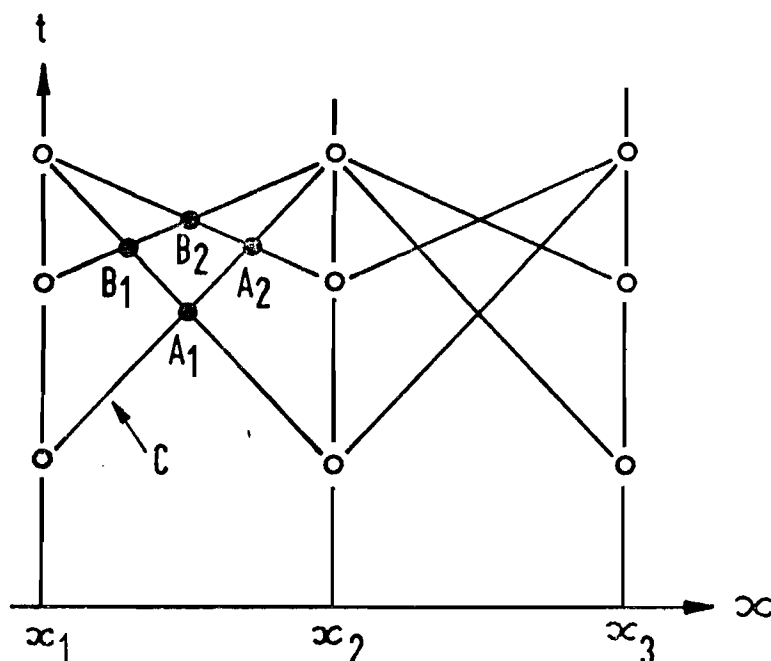


Fig. (11.3) Characteristic curves of a two circuit transmission line

Since the characteristic curves describe propagation of disturbances in the  $(x, t)$  plane, the integral:

$$\int_{x_1}^{x_2} dx \quad \text{along } C = \int_{x_1}^{A_1} dx + \int_{A_1}^{A_2} dx + \int_{A_2}^{x_2} dx$$

The function approximations discussed previously can be applied with the same accuracy as in the single circuit line only if the values of the variables are known at all the points of intersection of the characteristic curves with each other. There is no easy way of achieving this. Direct application of the method of characteristics to the general multi-circuit line will therefore be less accurate than for

the single circuit case. The extent of this cause of error depends on the solution grid distance increment  $\Delta x$ . Its effect can be seen in section 11.1.6).

If the condition for exact integration along a characteristic curve (section 11.1.3)) is present, the line may be solved for directly between its boundaries as in Fig. (11.4). This is because the integrals are defined entirely in terms of the initial and final values of the signals. For such lines an exact solution is obtained.

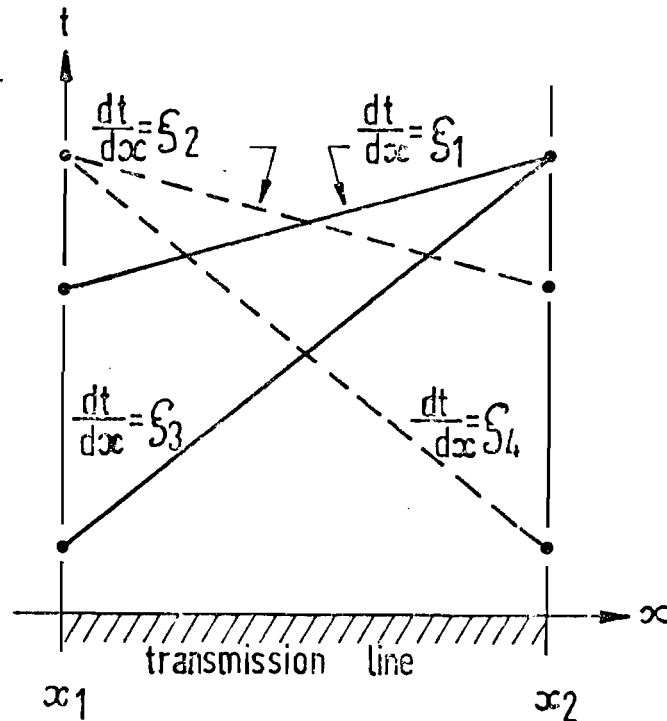


Fig. (11.4) Two circuit linear transmission line - characteristic curves between boundaries

#### 11.1.6) Example Solutions for a Two Circuit Symmetrical Transmission Line

A number of solutions have been carried out using direct application of the method of characteristics. These are compared with an exact result obtained using CARSON'S (1926) solution in conjunction with modal components (section

11.2.2)). The transmission line used has the profile given in Fig. (11.5) which corresponds to the New Zealand H.V.D.C. transmission line between Benmore and Fighting Bay.

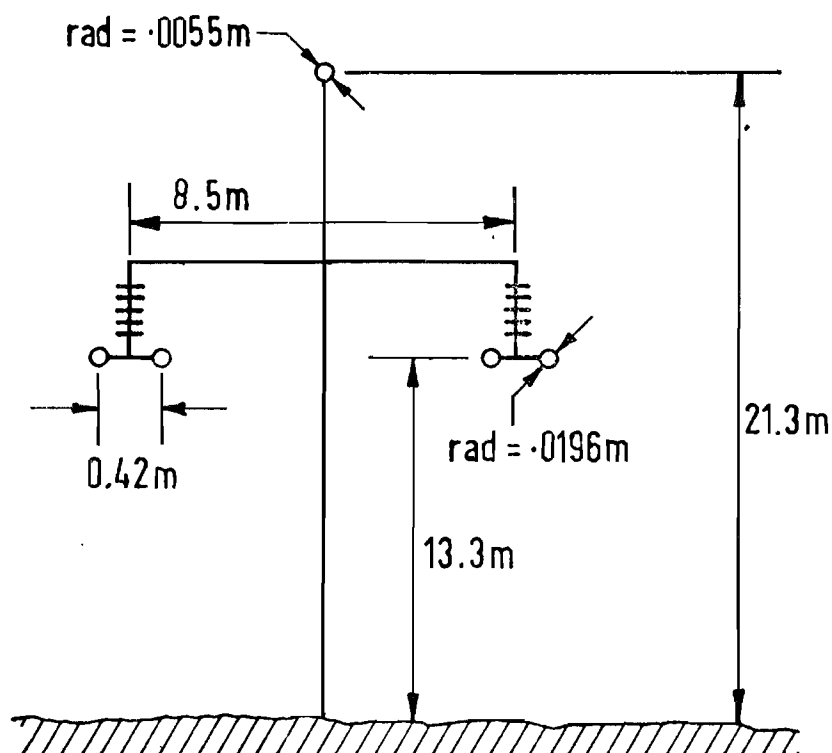


Fig. (11.5) Example transmission line - profile

The line capacitance coefficients were obtained using Maxwell's Potential coefficients with the overhead earth wire assumed to be at zero potential. The line inductance coefficients were obtained assuming a current image plane at a depth of 500 metres. This corresponds to steady state conditions of 414 Hz with an earth resistivity of 1000 ohm metres (WAGNER, 1933) and partially models the effect of a non-perfectly conducting ground. In this case, however, the depth of 500 metres was not chosen specifically to model ground conduction effects but to provide a separation between



modal components adequate for illustration purposes. A line conductor resistance of  $2 \times 10^{-5}$  ohms/metre was used and a constant ground return resistance of  $1.42 \times 10^{-4}$  ohms/metre was assumed. In each of the calculations the transmission line was assumed initially at rest. Line circuit 1 was energized with a unit step of voltage while line circuit 2 was held at zero potential (Fig. (11.13)). The line voltages and currents were computed at the line terminals and at a distance of 500 km from the terminals.

(a) Propagation time errors

The effect of solution-grid characteristic-grid mismatch is shown in Fig. (11.6). A solution distance increment of 100 km and a solution time increment of  $3.456 \times 10^{-5}$  seconds was used. This gave a solution grid which corresponded to the characteristic grid of the faster propagation component but which did not match the characteristic grid of the slower component (Fig. (11.2)). As a consequence there is no propagation time error for the faster component but a significant error for the slower component. In Fig. (11.7) the solution time increment was reduced to a quarter of that used in Fig. (11.6) with a subsequent reduction in propagation time error.

(b) Line current function approximations

In Fig. (11.7) an exponential current function approximation was used. In Fig. (11.8) a constant current function approximation was used. The differences between the two solutions are small. Less than 0.1% on the first component wavefront and 2% on the second component wavefront. The two approximations followed the same trends as in section 10) with the exponential approximation giving a line current that

tended high and voltages that were higher than the constant current approximation (see Tables (11.1) and (11.2)).

(c) Errors arising from incomplete solution along a characteristic curve

These errors are caused from not knowing the values of the line variables at all intersections of the characteristic curves (Fig. (11.3)). The effect of these errors is clearly seen in the line terminal current solution. In Figs (11.10) and (11.11) the line terminal currents are computed over a period of 10 milliseconds using a solution time increment of 1 millisecond, and a solution distance increment corresponding to propagation of the faster component. The solution characteristic propagation grids used corresponded to those shown in Fig. (11.9). It is seen that the line terminal currents will be stepped in pairs of increments as only every second solution point is affected by the component propagation. Note also that the characteristic grid of the faster component impinges on the boundary every second solution point whereas the characteristic grid of the slower component impinges on the boundary every fourth solution point. This gives most accuracy every fourth solution point and accounts for the stepped and oscillatory nature of the solution. The effects of this error can be reduced by reducing the solution grid size as shown in Figs (11.10) and (11.11) where the step size is reduced to a quarter of that originally used.

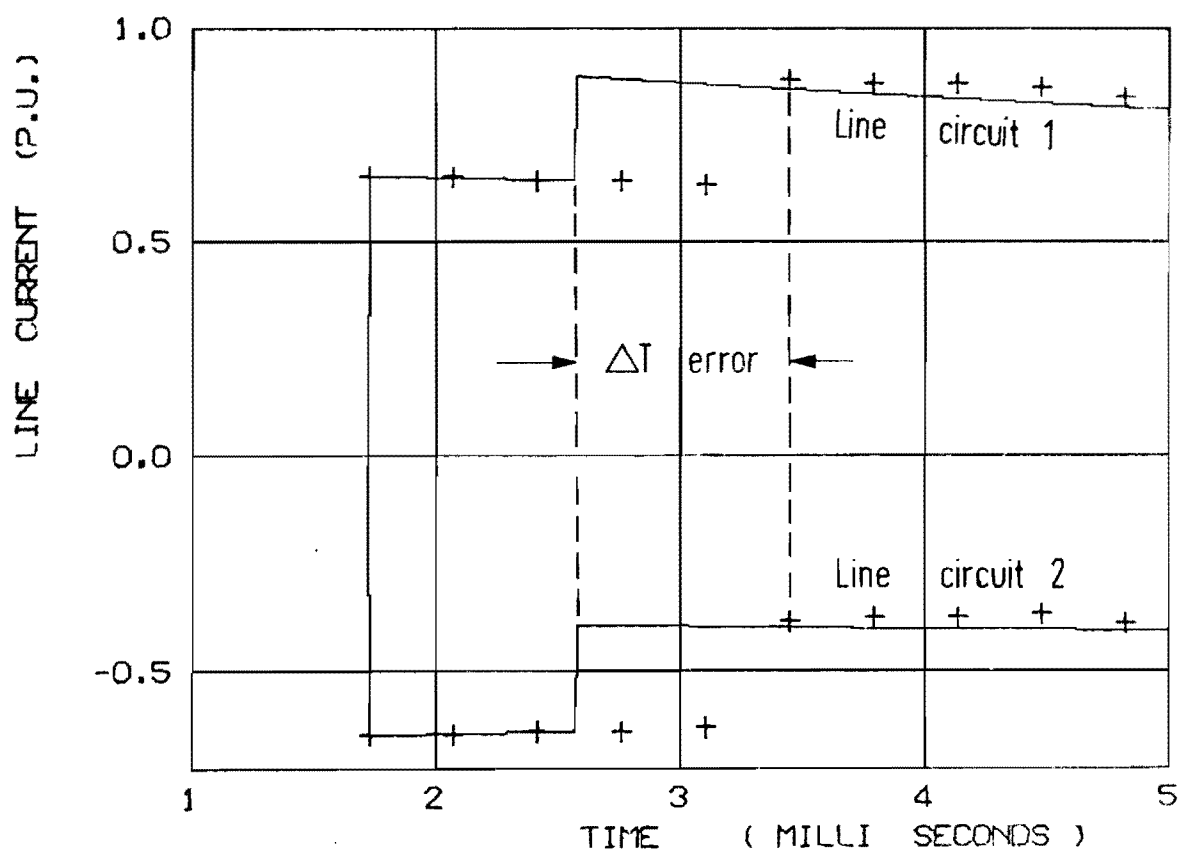
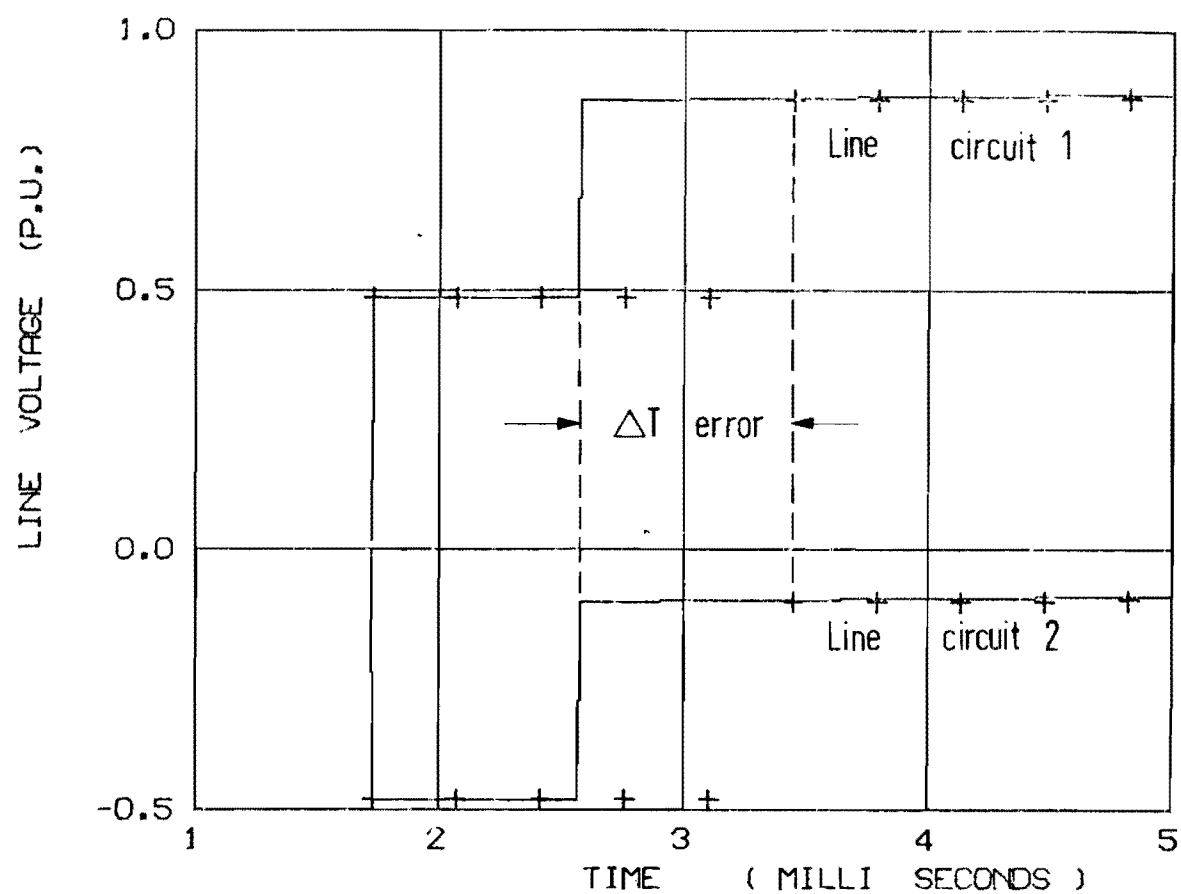


Fig. (11.6) Direct application of M.O.C. to a two circuit line. Solution at  $X = 500$  km. Exponential current approximation.

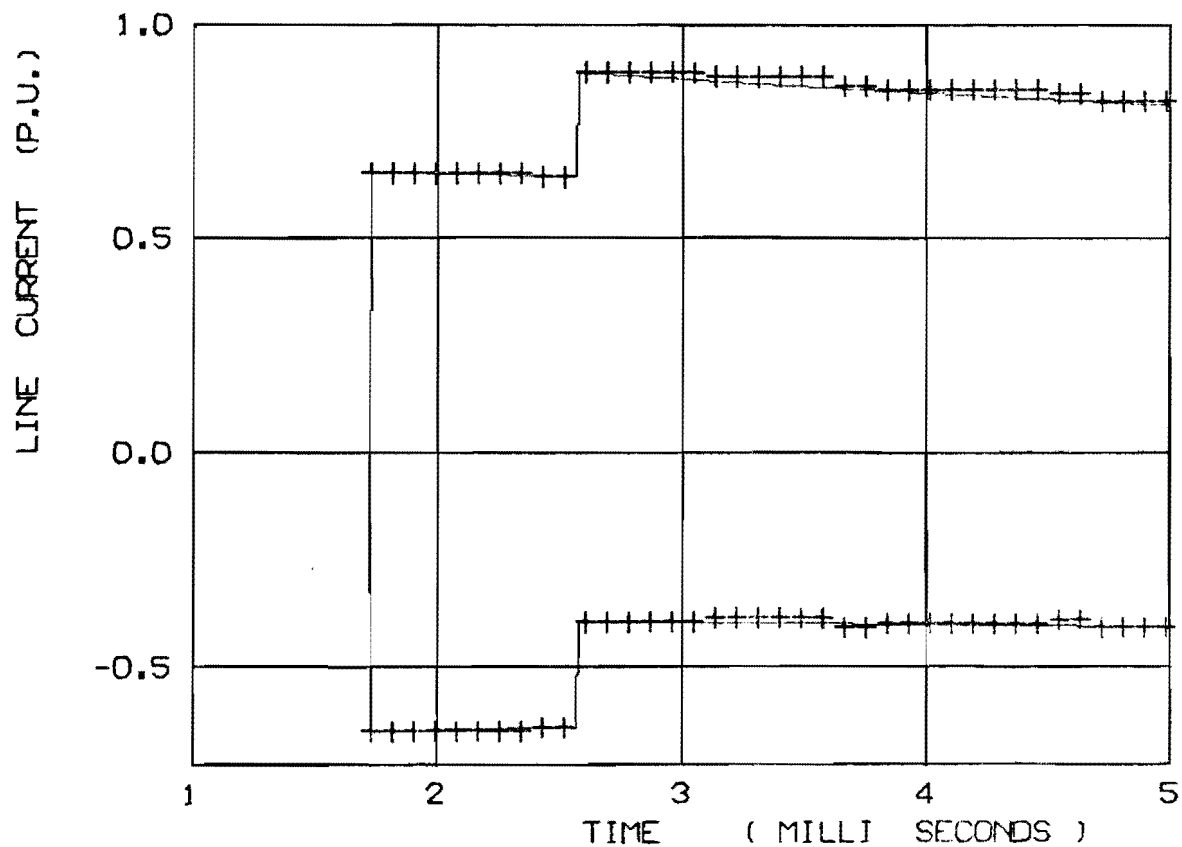
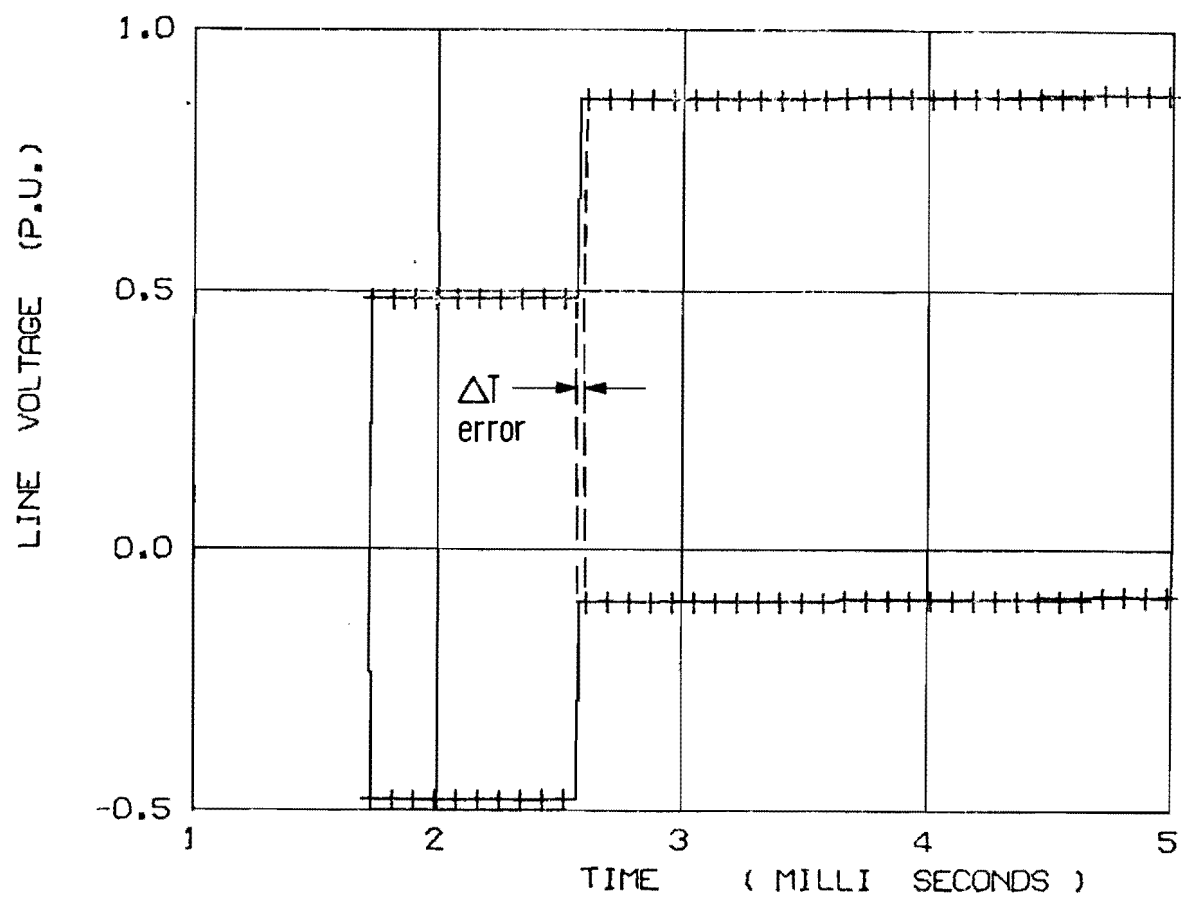


Fig. (11.7) Direct application of M.O.C. to a two circuit line. Solution at  $X = 500$  km. Exponential current approximation.

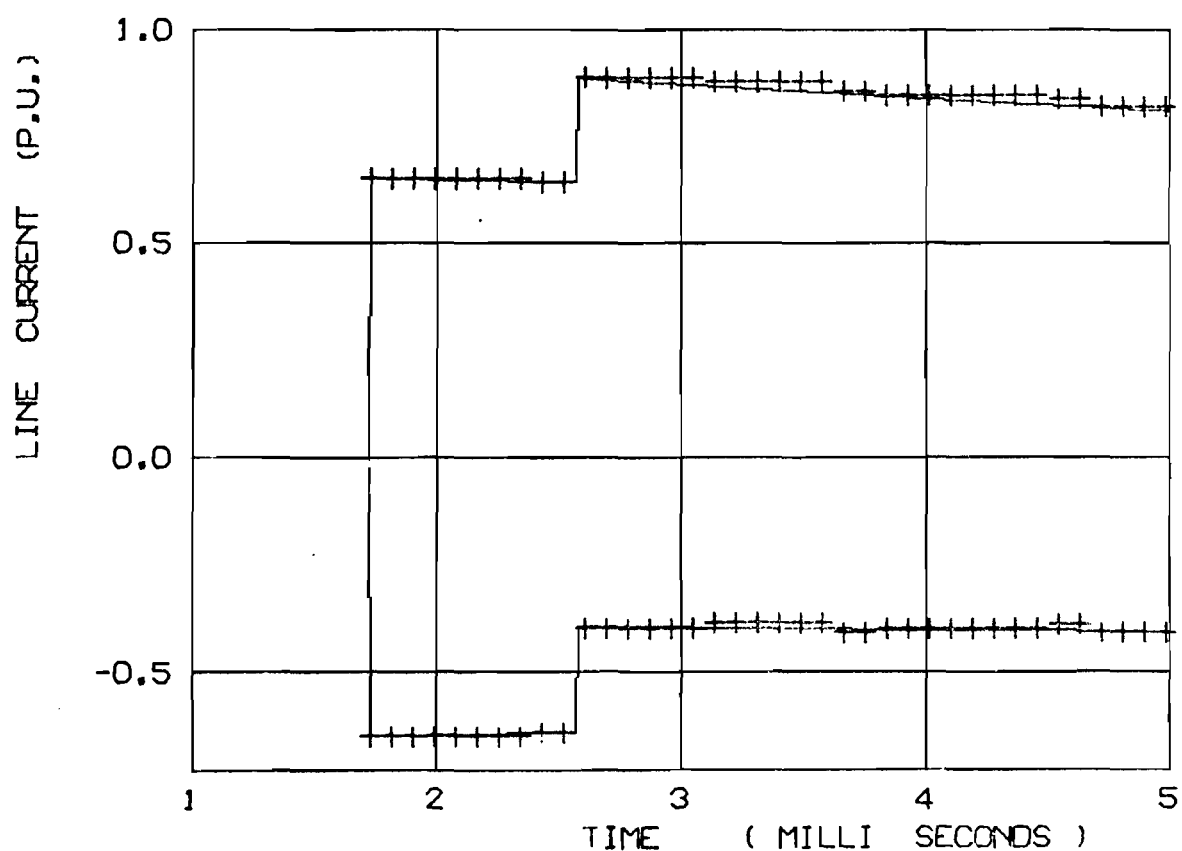
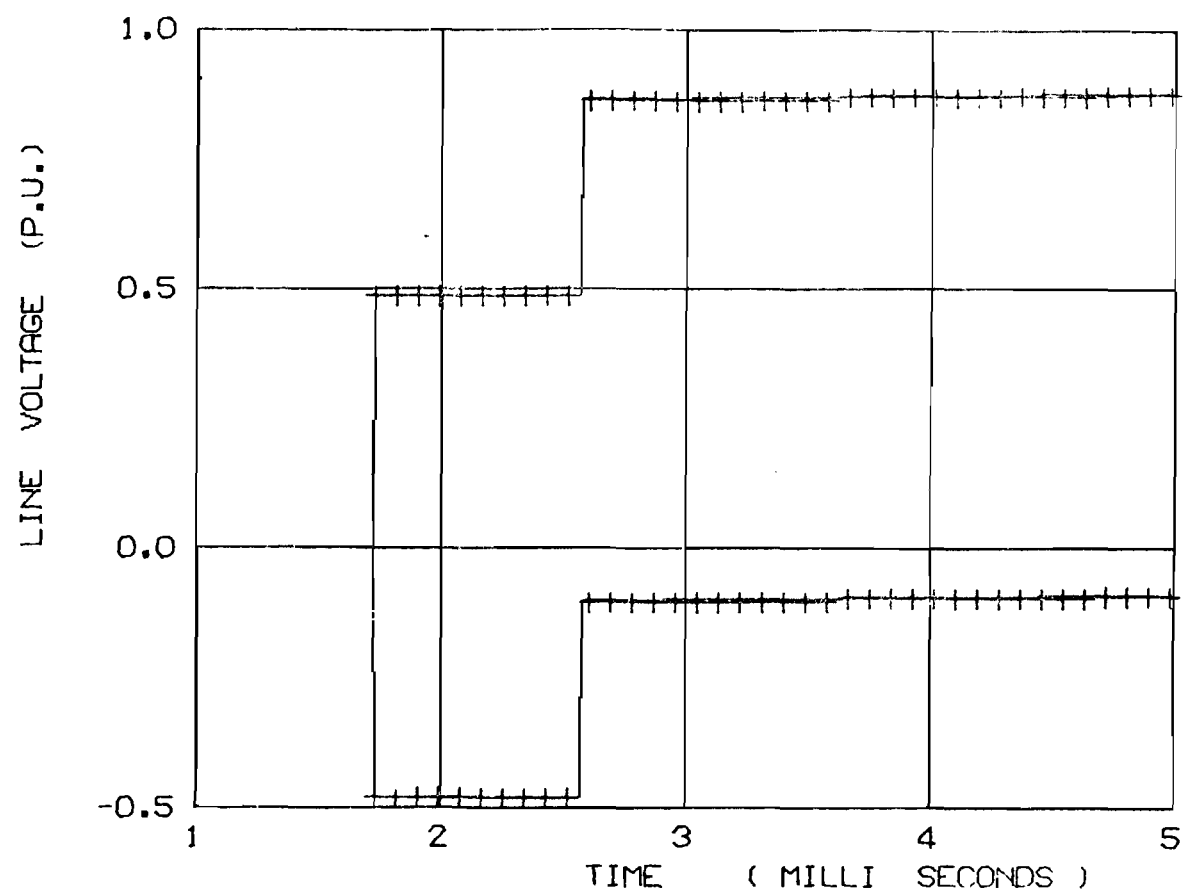


Fig. (11.8) Direct application of M.O.C. to a two circuit line. Solution at  $X = 500$  km. Constant current approximation.

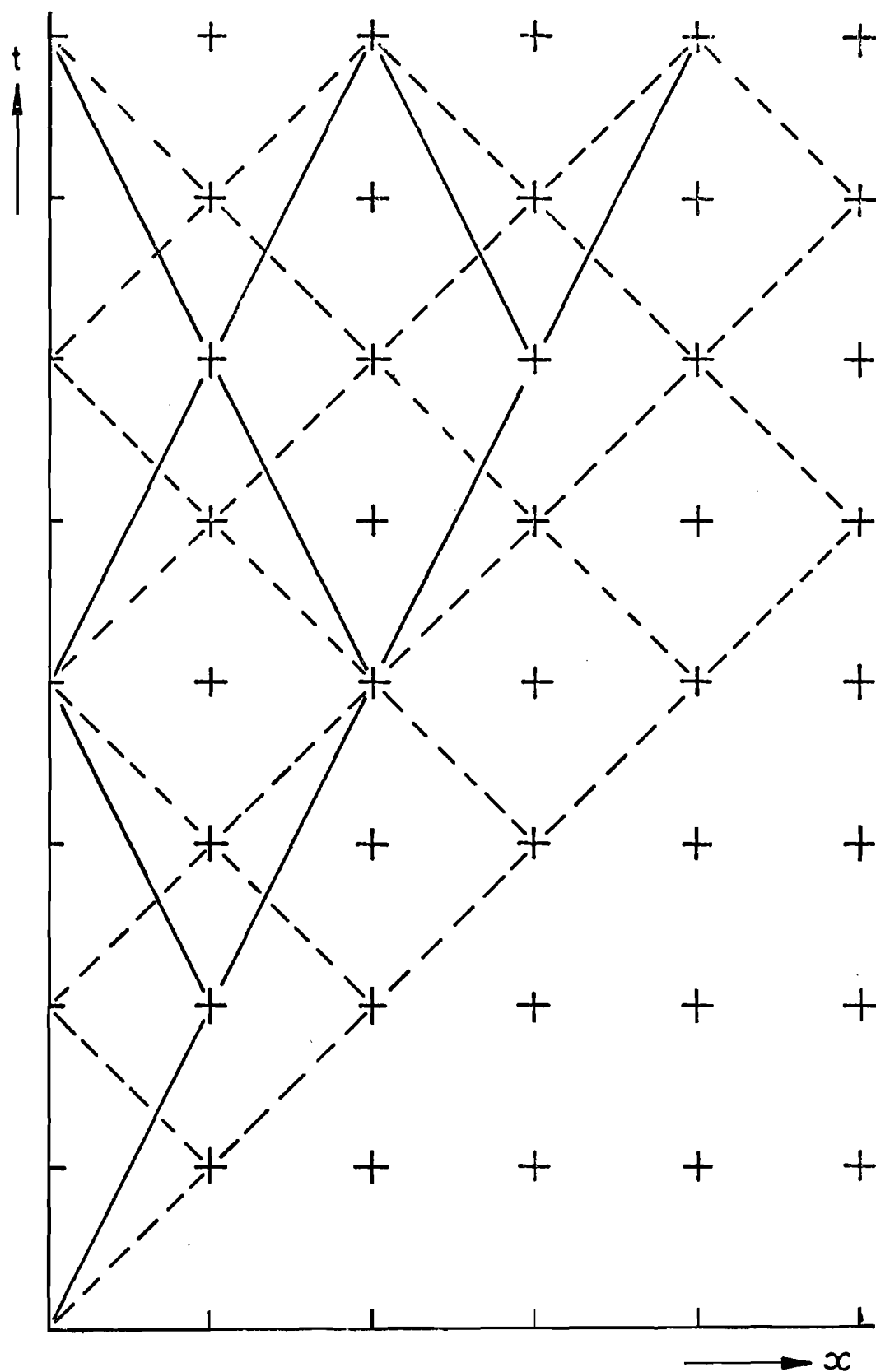


Fig. (11.9) Solution grid used in example 11.1.5) (c)

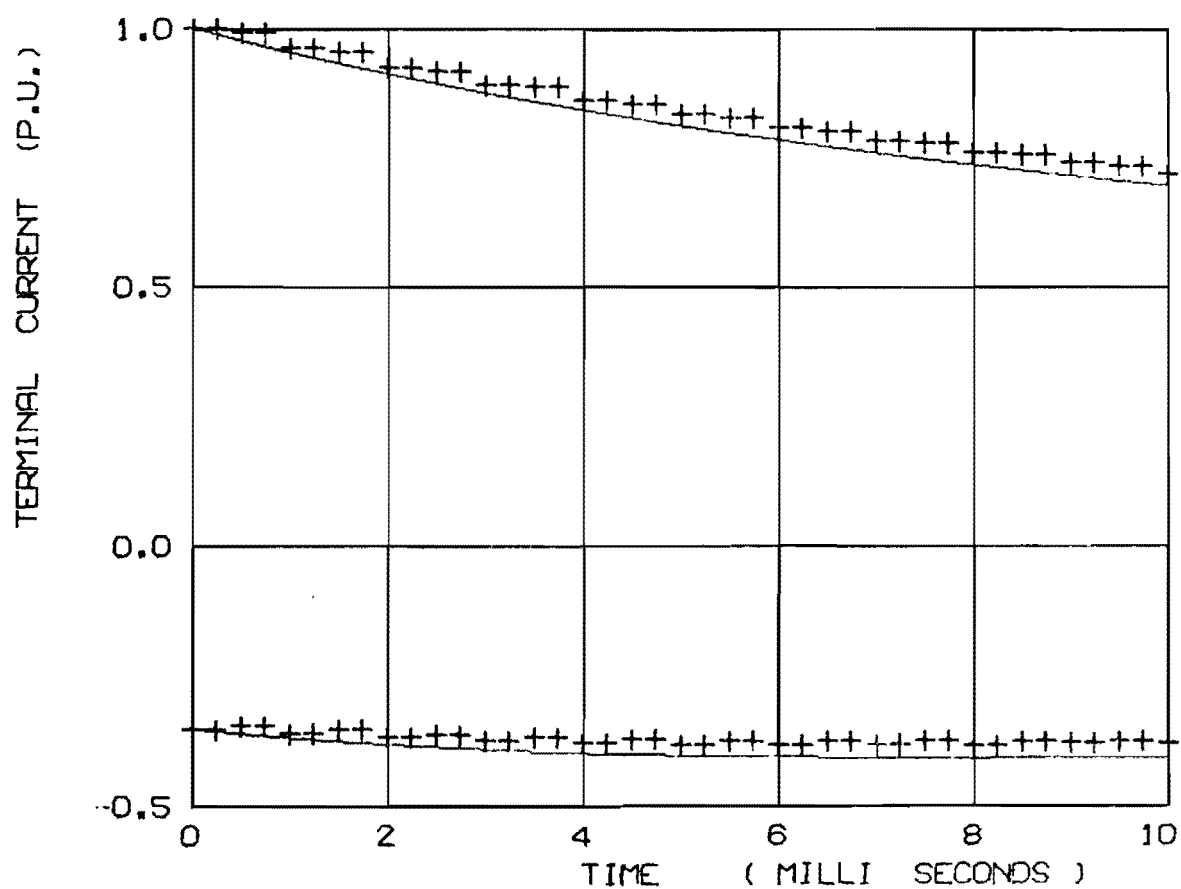
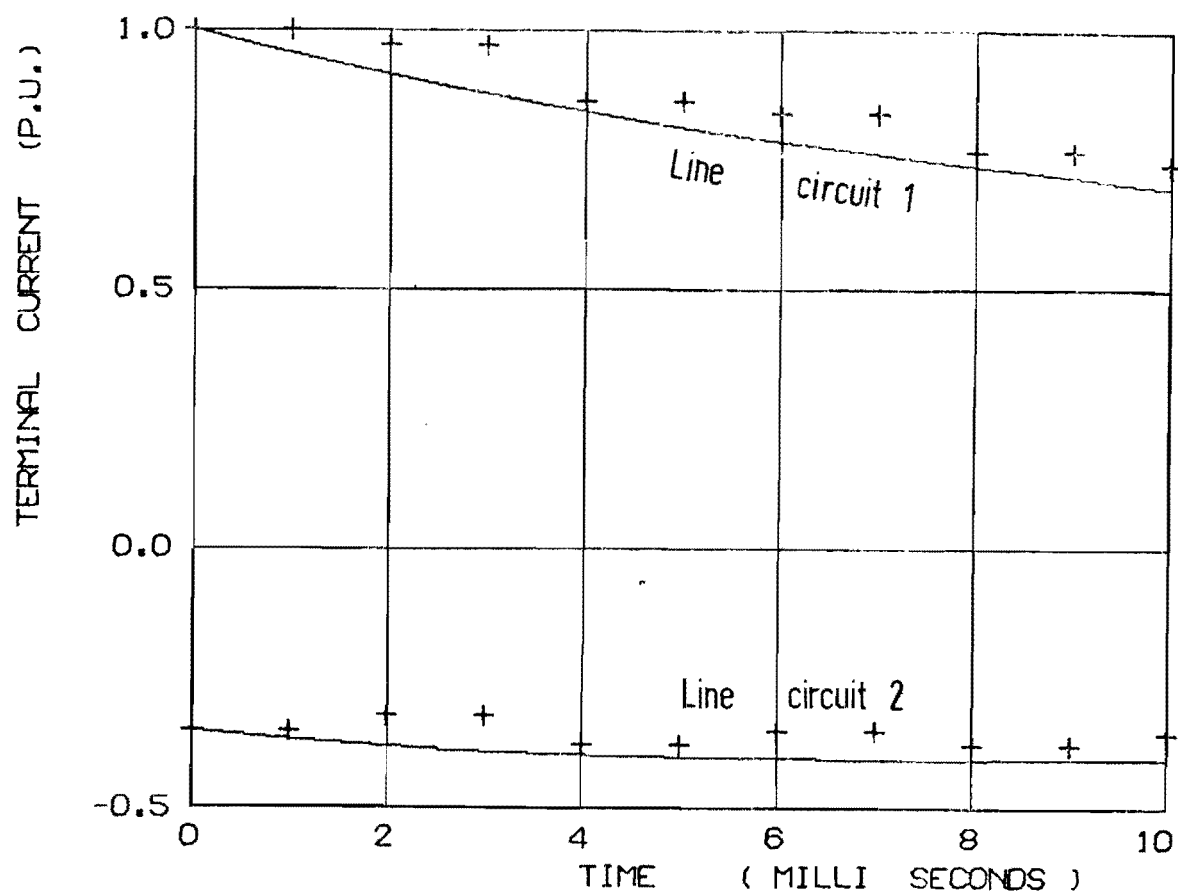


Fig. (11.10) Direct application of M.O.C. to a two circuit line. Exponential current approximation.

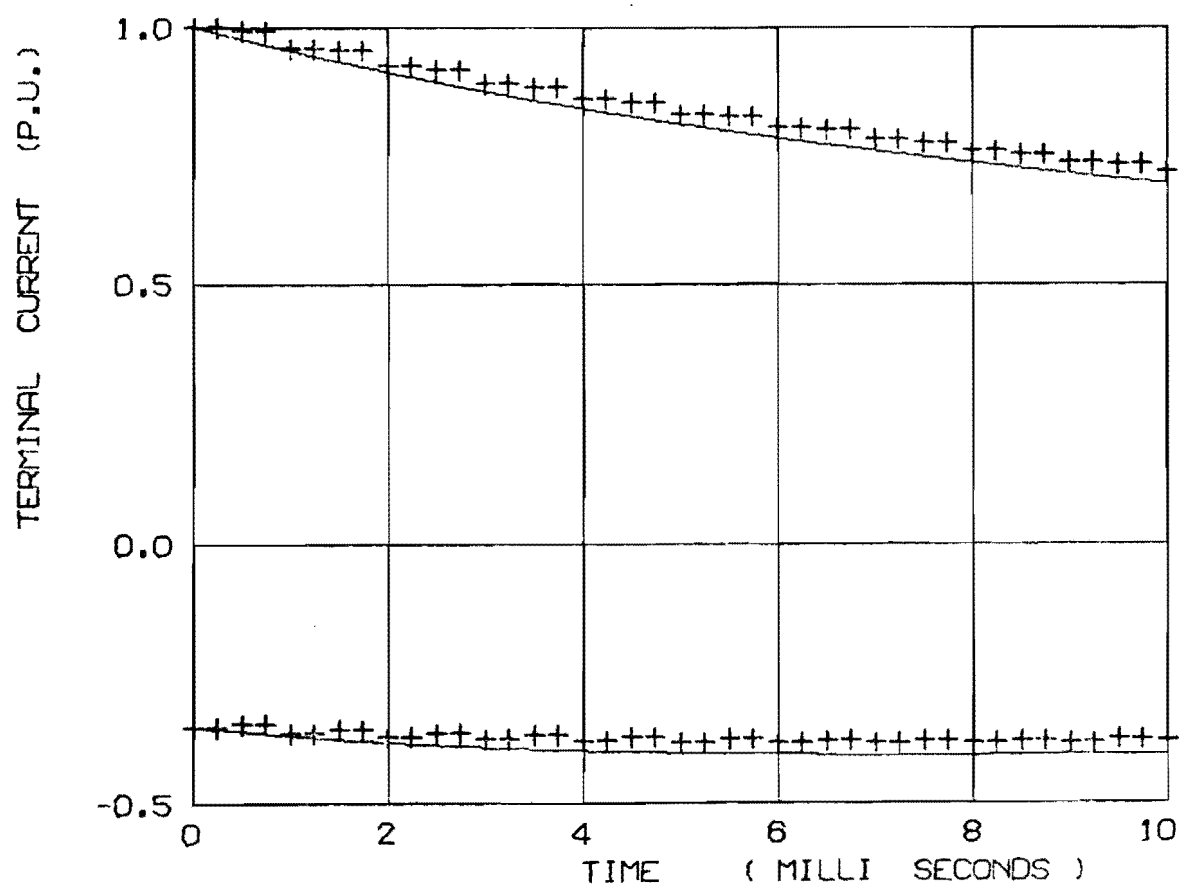
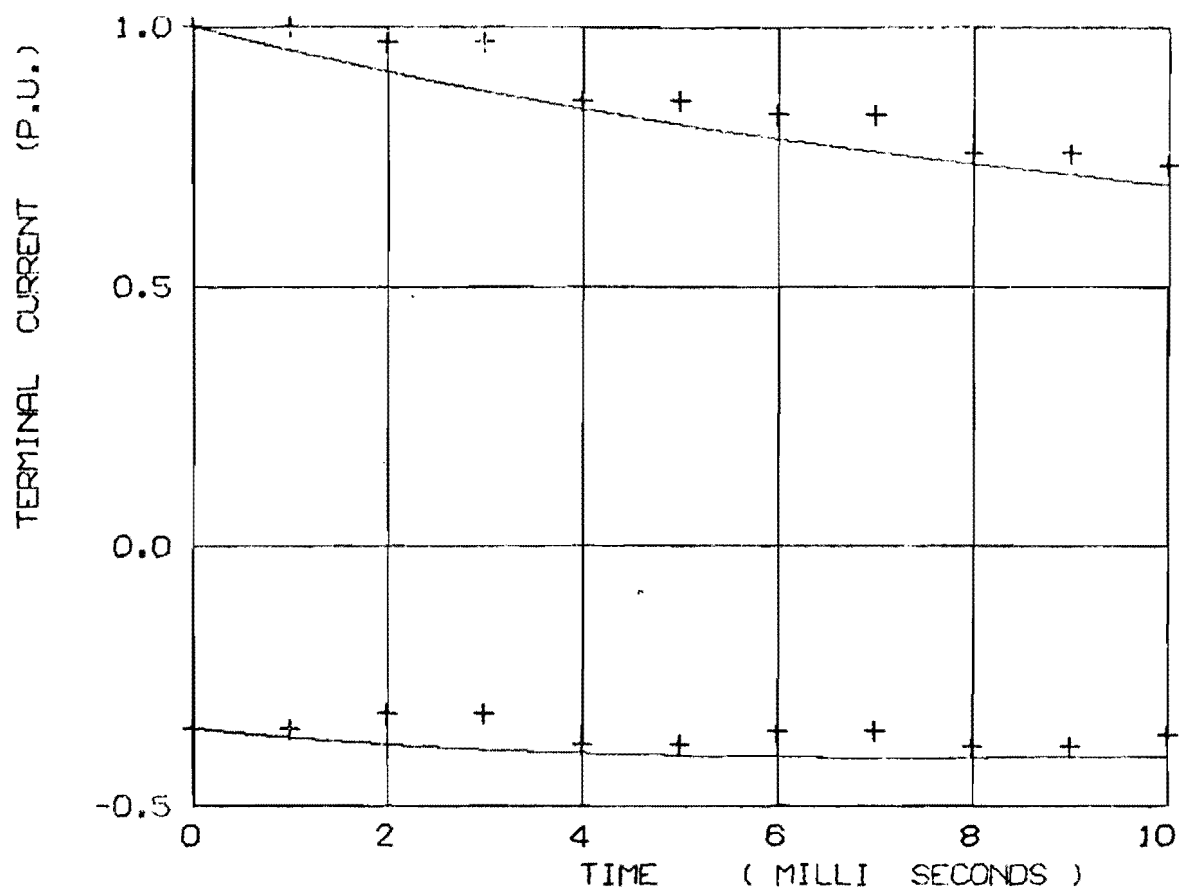


Fig. (11.11) Direct application of M.O.C. to a two circuit line. Constant current approximation.



(d) Discussion of errors

The errors outlined above are inherent in the direct application of the method of characteristics to the solution of propagation problems on general lossy multiple circuit transmission lines. They can be reduced by suitably adjusting the solution grid sizes. Of the errors considered, those arising from incomplete solution along a characteristic are most serious. This source of error can be reduced by using the indirect method of section 11.2.3) and eliminated by using the indirect methods of sections 11.2.2) and 11.2.4). However, these indirect methods cannot always be applied.

11.2) APPLICATION OF MATRIX METHODS - MODAL COMPONENTS

The application of matrix methods to diagonalize the T.L.E.s was introduced in section 4.2.1)(b). It was pointed out in section 4.7.2) that this method of solution has only limited application in the time domain. Almost invariably it is preferred in those cases where it can be used. One example of a line to which the method can be applied is the mutually coupled two circuit symmetrical transmission line and this will be used to illustrate the application of this technique.

11.2.1) Component Signals of a Two Circuit Symmetrical Line

The T.L.E.s of such a line are:

$$-\frac{\partial}{\partial x} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}_L = \begin{bmatrix} L_s & L_m \\ L_m & L_s \end{bmatrix} \frac{\partial}{\partial t} \begin{Bmatrix} i_1 \\ i_2 \end{Bmatrix}_L + \begin{bmatrix} R_s & R_m \\ R_m & R_s \end{bmatrix} \begin{Bmatrix} i_1 \\ i_2 \end{Bmatrix}_L \quad (11.18)$$

$$-\frac{\partial}{\partial x} \begin{Bmatrix} i_1 \\ i_2 \end{Bmatrix}_L = \begin{bmatrix} C_s & C_m \\ C_m & C_s \end{bmatrix} \frac{\partial}{\partial t} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}_L + \begin{bmatrix} G_s & G_m \\ G_m & G_s \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}_L$$

The line variables are linearly transformed to a set of component variables. The transformation is carefully chosen so that the resulting component equations contain no mutual coupling. Each component then appears to be travelling on a single circuit transmission line and can be solved accordingly. The line variables can be recovered from the component solution by inverse transformation. Using the approach of (WEDEPOHL, 1963) the transformation for a two circuit symmetrical line can be shown to be:

$$\{v\}_L = [s]\{v\}_C \quad (11.19)$$

$$\{i\}_L = [s]\{i\}_C$$

$$\text{where } [s] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The inverse is:

$$\{v\}_C = [s]^{-1}\{v\}_L \quad (11.20)$$

$$\{i\}_C = [s]^{-1}\{i\}_L$$

$$\text{where } [s]^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Using (11.19) in (11.18) gives the transformed equations:

$$-\frac{\partial}{\partial x} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}_c = \begin{bmatrix} L_s + L_m & 0 \\ 0 & L_s - L_m \end{bmatrix} \frac{\partial}{\partial t} \begin{Bmatrix} i_1 \\ i_2 \end{Bmatrix}_c + \begin{bmatrix} R_s + R_m & 0 \\ 0 & R_s - R_m \end{bmatrix} \begin{Bmatrix} i_1 \\ i_2 \end{Bmatrix}_c \quad (11.21)$$

$$-\frac{\partial}{\partial x} \begin{Bmatrix} i_1 \\ i_2 \end{Bmatrix}_c = \begin{bmatrix} C_s + C_m & 0 \\ 0 & C_s - C_m \end{bmatrix} \frac{\partial}{\partial t} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}_c + \begin{bmatrix} G_s + G_m & 0 \\ 0 & G_s - G_m \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}_c .$$

where  $v_{c_1} = \frac{1}{2}(v_{L1} + v_{L2})$ ;

$i_{c_1} = \frac{1}{2}(i_{L1} + i_{L2})$  is component one, say.

$v_{c_2} = \frac{1}{2}(v_{L1} - v_{L2})$ ;

$i_{c_2} = \frac{1}{2}(i_{L1} - i_{L2})$  is component two, say.

It is seen that component one has a surge impedance of  $Z_1 = \sqrt{(L_s + L_m)/(C_s + C_m)}$ , sees a series resistance of  $(R_s + R_m)$ , a shunt conductance of  $(G_s + G_m)$  and has a velocity of propagation  $a_1 = 1.0/\sqrt{(L_s + L_m)(C_s + C_m)}$  which is slower than component two which has  $a_2 = 1.0/\sqrt{(L_s - L_m)(C_s - C_m)}$  etc. No coupling terms are present in equations (11.21) and the two components can be solved separately. The line variables can be recovered from (11.19) giving:

$$v_{L1} = v_{c_1} + v_{c_2}; \quad v_{L2} = v_{c_1} - v_{c_2}; \text{ etc.}$$

Care must be exercised when solving at points of discontinuity. A component wave impinging on a discontinuity can generate other component signals. Consider Fig. (11.12) in which a two circuit transmission line has line circuit 1 terminated in a short circuit and line circuit 2 terminated in an open circuit.

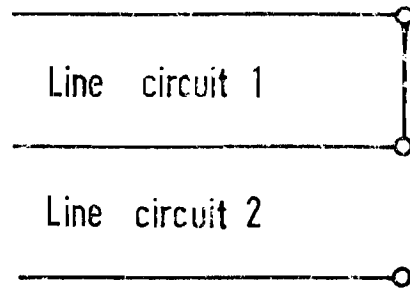


Fig. (11.12) Two circuit unbalanced line termination

At the line terminal we have:

$$\{v_r\}_L = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \{v_i\}_L \quad (11.22)$$

where subscript r denotes reflected

" " i denotes incident

Transforming (11.22) into component variables using (11.19) gives:

$$\{v_r\}_c = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \{v_i\}_c \quad (11.23)$$

from which it is seen that an incident component one wave generates a reflected component two wave and vice-versa. Because discontinuities are more easily visualized in terms of line variables and because line variables enable the solution of non-linear problems, it is usual practice to refer to line rather than component variables at points of discontinuity.

### 11.2.2) Use of CARSON'S (1926) Solution for Components

Being able to break a multiple circuit line into a number of uncoupled single circuit lines offers the possibility of exact solutions. If the component solutions can be solved for exactly, a total exact solution can be obtained. Provided that the lines are terminated properly CARSON'S (1926) solution can be applied directly to the components yielding an exact solution.

As an example, consider a mutually coupled two circuit line which is initially at rest. A voltage,  $E$ , is applied to line circuit 1 while line circuit 2 is short circuited (Fig. (11.13)).

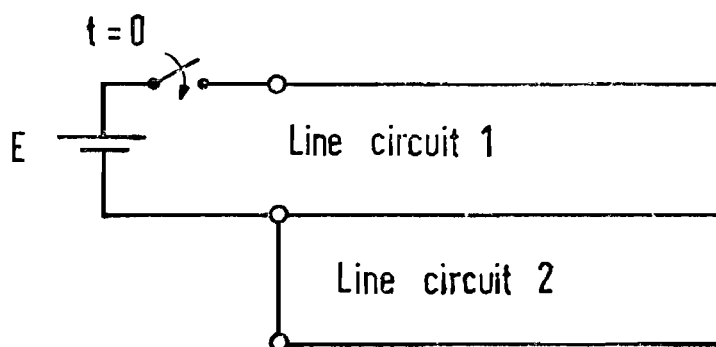


Fig. (11.13) Terminal conditions for example solution using CARSON'S solution for a single circuit line.

The component equations are given in (11.21). The component initial conditions are obtained from equations (11.20).

$$v_{C_1} = \frac{1}{2}(V_{L1} + v_{L2}) = \frac{1}{2}E$$

$$v_{C_2} = \frac{1}{2}(V_{L1} - v_{L2}) = \frac{1}{2}E$$

Thus both component one and component two have a step voltage of  $\frac{1}{2}E$  impressed on them. These two components propagate at different velocities (Fig. (11.14)), have different

attenuation and different surge impedances.

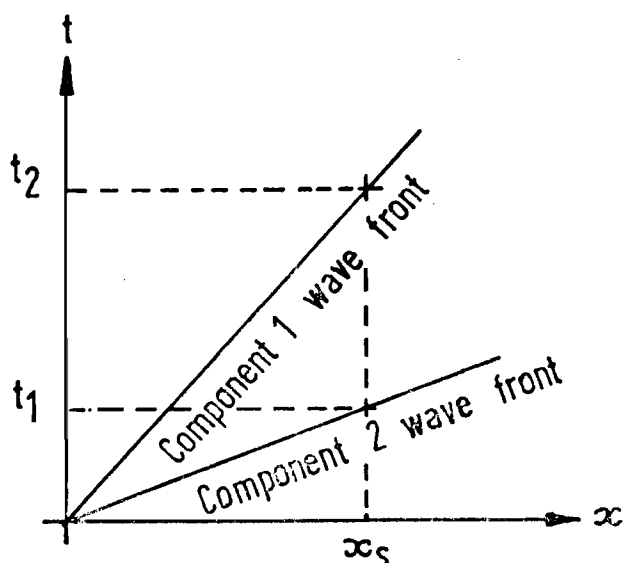


Fig. (11.14)

At time  $t_1$ , component two arrives at  $x_s$  and gives rise to line voltages of opposite polarity. At time  $t_2$  component one arrives at  $x_s$  and gives rise to voltages of similar polarity. At point  $x_s$  the line signals bear little resemblance to the applied signal. This is a characteristic feature of multiple circuit transmission lines.

In this problem the component terminal conditions were compatible with CARSON'S solution which could be applied directly yielding exact solutions. These have been computed for a semi-infinite line with  $E$  a unit voltage and are shown in Fig. (11.15). The transmission line used was the same as that used in section 11.1.6) and this result provides the exact solution used throughout section 11).

The limitations of applying solutions such as CARSON'S solution are illustrated by the following example. Here the complexity of the problem is considerably increased by having

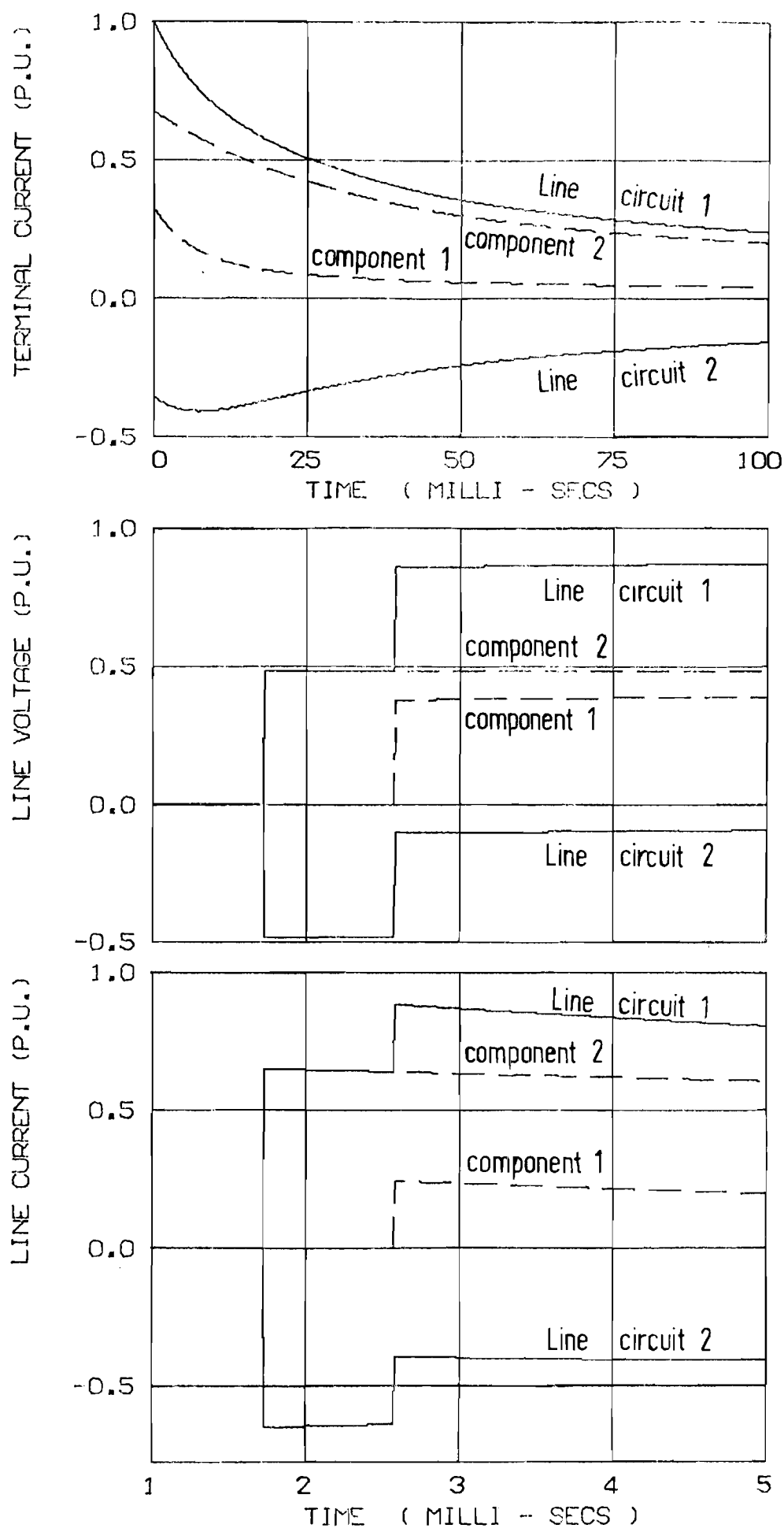


Fig. (11.15) Exact solution for two circuit line using modal components and CARSON'S (1926) solution. Signal applied as in Fig. (11.13).

the terminal of line circuit 2 open circuited (Fig. (11.16)).

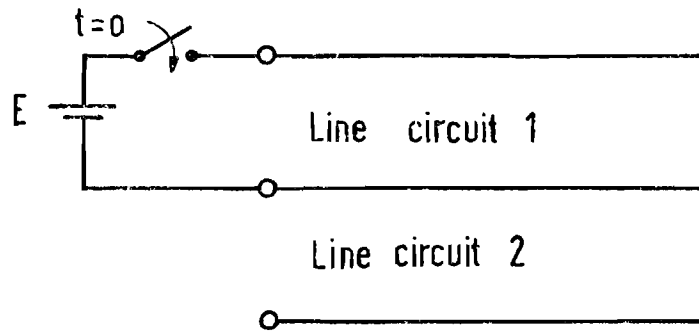


Fig. (11.16)

The line terminal conditions are now  $v_{L1} = E$  and  $i_{L2} = 0$ . The component terminal conditions are:

$$\left. \begin{aligned} v_{C1} &= \frac{1}{2}E + \frac{1}{2}v_{L2} \\ v_{C2} &= \frac{1}{2}E - \frac{1}{2}v_{L2} \end{aligned} \right\} \quad (11.24)$$

$$\left. \begin{aligned} i_{C1} &= \frac{1}{2}i_{L1} \\ i_{C2} &= \frac{1}{2}i_{L1} \end{aligned} \right\} \quad (11.25)$$

or  $i_{C1} = i_{C2}$

Except for lines with dispersionless components CARSON'S solution cannot now be directly applied as there is no longer a step energization of the components. Further, to obtain the component terminal conditions another relation between the variables  $v_{C1}$ ,  $v_{C2}$  and  $v_{L2}$  must be found. For lines with dispersionless components this can be found using equations (11.25) and the fact that voltage and current are always related by the surge impedance.



$$i_{C_1} = i_{C_2} = \frac{v_{C_1}}{Z_1} = \frac{v_{C_2}}{Z_2}$$

or  $\frac{E + v_{L2}}{Z_1} = \frac{E - v_{L2}}{Z_2}$

giving

$$v_{C_1} = E \frac{Z_1}{Z_1 + Z_2} \quad (11.26)$$

$$v_{C_2} = E \frac{Z_2}{Z_1 + Z_2}$$

For the general line the component voltages and currents are related by their surge impedances only at the toe of a propagating wave and equations (11.26) apply only at the instant that the voltage  $E$  is connected. An approximate solution can be obtained by assuming the terminal signals to be the sum of time displaced step functions, using CARSON'S solution for each step and obtaining the total response by superposition. Here the terminal step voltages are determined so that at the instant of application conditions (11.25) and (11.24) are simultaneously satisfied. This gives for steps applied after  $t = 0$ :

$$\Delta v_{C_1} = \frac{I_{C_2} - I_{C_1}}{\left(\frac{1}{Z_1} - \frac{1}{Z_2}\right)}$$

$$\Delta v_{C_2} = -\Delta v_{C_1}$$

where  $I_C$  is the total component current at the instant before the step is applied.

#### Comment

CARSON'S solution, by virtue of its restricted nature, (applies only for a step function of voltage signal), could

not be applied directly to the above problem. As the solution was not suited to the problem the problem had to be made suitable for the solution. Solution using the method of characteristics would not have posed these difficulties as its requirements, point values of the components in time past, are readily deduced from the solution already acquired. Furthermore, solution at given points can readily be expressed in terms of line variables, greatly simplifying boundary condition specification. The method of characteristics is thus more flexible than CARSON'S solution in its direct application to problems expressed in terms of modal components.

### 11.2.3) Combining the Method of Characteristics and Modal Components

Component propagation can be solved by applying the method of characteristics. Traditionally this approach has been restricted to the situation where the transmission line components are dispersionless. For this situation the components can be solved directly between the line boundaries and exact solutions can be obtained.

Consider the  $(x,t)$  propagation diagram of a two circuit dispersionless line (Fig. (11.17)). To find the total solution at  $x_0$  at time  $t_3$  both components need be known. The component characteristic equations at this point are:

$$\begin{aligned} (v_{c_2} - Z_2 \cdot i_{c_2})(x_0, t_3) &= f(v_{c_2}, i_{c_2})(x_1, t_2) \\ (v_{c_1} - Z_1 \cdot i_{c_1})(x_0, t_3) &= f(v_{c_1}, i_{c_1})(x_1, t_1) \end{aligned} \quad (11.27)$$

where  $f(v_k, i_k) = \beta_k (v_k - z_k i_k)$

$\beta_k$  is an attenuation factor  $\leq 1$  (section 9.1))

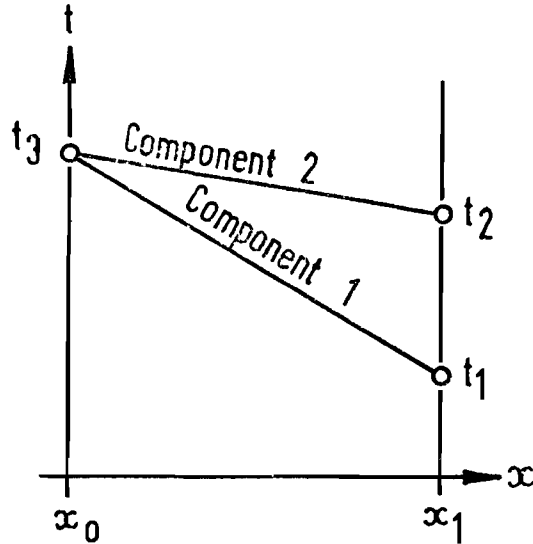


Fig. (11.17)

Equations (11.27) can be solved by specifying two further component relations at the line terminal. This, as shown in section 11.2.2), can sometimes be difficult to do. Usually however, terminal conditions are readily available in terms of line quantities and so it is convenient to convert equations (11.27) to these. Using equations (11.20) gives:

$$\begin{aligned} (v_{L1} - v_{L2} - z_2 \cdot i_{L1} + z_2 \cdot i_{L2})(x_0, t_3) &= 2f(v_{c_2}, i_{c_2})(x_1, t_2) \\ (v_{L1} + v_{L2} + z_1 \cdot i_{L1} + z_1 \cdot i_{L2})(x_0, t_3) &= 2f(v_{c_1}, i_{c_1})(x_1, t_1) \end{aligned} \quad (11.28)$$

With two more relations specified between line variables at  $(x_0, t_3)$  equations (11.28) can be solved.

It is a small step to convert equations (11.28) into equations identical to those obtained by applying the method of characteristics directly to the multiple circuit transmission line. As it is more direct and as both

approaches give rise to an exact solution, applying the method of characteristics directly to the multiple circuit transmission line (section 11.1.2)) is to be preferred to separating the problem into components and then solving. Traditionally however, the component approach has been used.

This approach can also be used when the components propagate with distortion although it is less satisfactory than for the dispersionless case. Approximate solutions for components would be obtained using the techniques of section 10). Here each component would be ascribed its own rectangular solution grid (corresponding to its particular characteristic grid) which may on account of accuracy require extra solution points between the line boundaries. Matching of the component grids at line boundaries will produce all the problems and errors outlined in section 11.1.5). Their effects, however, will be smaller than those illustrated in section 11.1.6) as only one solution distance increment is affected per line boundary.

#### Comment

From the above it is seen that this approach is most suited to a solution which can be applied directly between any two points on a transmission line. Thus far, the method of characteristics has achieved this only where dispersionless transmission is present or where the errors involved in assuming such an application of an approximate solution can be tolerated. The availability of a line point to line point solution would eliminate errors due to incomplete solution along a characteristic and would hold propagation time errors to a maximum of half the solution

time increment. For these reasons such a solution has been developed in section 12). Here the method of characteristics is formulated between any two points on a general single circuit transmission line using the transmission line's impulse responses. The application of this to a multiple circuit transmission line using the above method is illustrated in section 11.2.4).

#### 11.2.4) Use of Line Point to Line Point Formulation of the Method of Characteristics

From section 11.2.3) it is apparent that the method of modal components is most suited for use with solutions that can be applied directly between any two points on a transmission line. In section 11.2.2) the desirability of using a solution which can easily handle boundary conditions is pointed out. In a line point to line point formulation of the method of characteristics we have such a solution. This solution is derived in section 12) and its application to a two circuit symmetrical line is considered here. In order to compare the performance of this solution with the other methods already considered the same problem will be solved as was solved in section 11.1.6). The line terminal currents and the line voltages and currents at  $X = 500$  km will be found.

For each component the terminal currents were found from:

$$(v - Z_c i)_{(0,n\Delta T)} = \Delta T \left\{ \sum_{k=1}^n [v_{(0,p\Delta T)} \times W^3_{(0,k\Delta T)} + Z_c i_{(0,p\Delta T)} \times W^4_{(0,k\Delta T)}] \right\} \quad (11.29)$$

$$v_{(0,t)} = \text{constant}$$

The line voltages and currents were found from:

$$\begin{aligned}
 (v + Z_c i)_{(x, n\Delta T)} &= (v + Z_c i)_{(0, n\Delta T - \frac{x}{a})} e^{-\frac{\rho x}{a}} \\
 &+ \Delta T \left\{ \sum_{k=k_1}^n [v_{(0, p\Delta T)} \times W_1(x, k\Delta T) \right. \\
 &\left. + Z_c i_{(0, p\Delta T)} \times W_2(x, k\Delta T) \right\} \quad (11.30)
 \end{aligned}$$

$$\begin{aligned}
 (v - Z_c i)_{(x, n\Delta T)} &= \Delta T \left\{ \sum_{k=k_1}^n [v_{(0, p\Delta T)} \times W_3(x, k\Delta T) \right. \\
 &\left. + Z_c i_{(0, p\Delta T)} \times W_4(x, k\Delta T) \right\}
 \end{aligned}$$

where  $p = n - k$  and  $k_1 = \frac{x}{a\Delta T}$

Equations (11.29) and (11.30) are written in a form in which they can be used on a digital computer. A comparison with equations (7.15) and (7.17) shows them to have separated the distorting and non-distorting components of propagation. The terms in the curly brackets are identically zero for dispersionless transmission (section 12.1)). The ease with which arbitrary transmission line boundary conditions can be specified is evident.

In the solution computed here, equations (11.29) and (11.30) were written in terms of the per unit system for a series resistance line and the appropriate per unit components were solved. The components were then combined to give the illustrated solutions. Functions  $W_1$ ,  $W_2$ ,  $W_3$  and  $W_4$  were calculated with numerical evaluation of the integrals to a maximum error of  $1 \times 10^{-5}$  over the range of integration (section 12.2.2)). The results shown in Fig. (11.18) and

(11.19) should be compared with Fig. (11.6) and (11.10) respectively. In these the exact solution is given by the continuous line and the method of characteristics solution at the points where it was computed by a '+'. It will be seen that the propagation time error has been reduced to  $< \frac{1}{2}\Delta T$  and the errors due to incomplete solution along a characteristic have been eliminated.

For small values of  $\Delta T$  very accurate solutions can be obtained. This is illustrated by comparing Tables (11.3) and (11.4). In both cases a solution time increment of  $\Delta T = 1.728 \times 10^{-5}$  seconds was used.

Tables (11.3) and (11.4) can also be compared with tables (11.1) and (11.2).

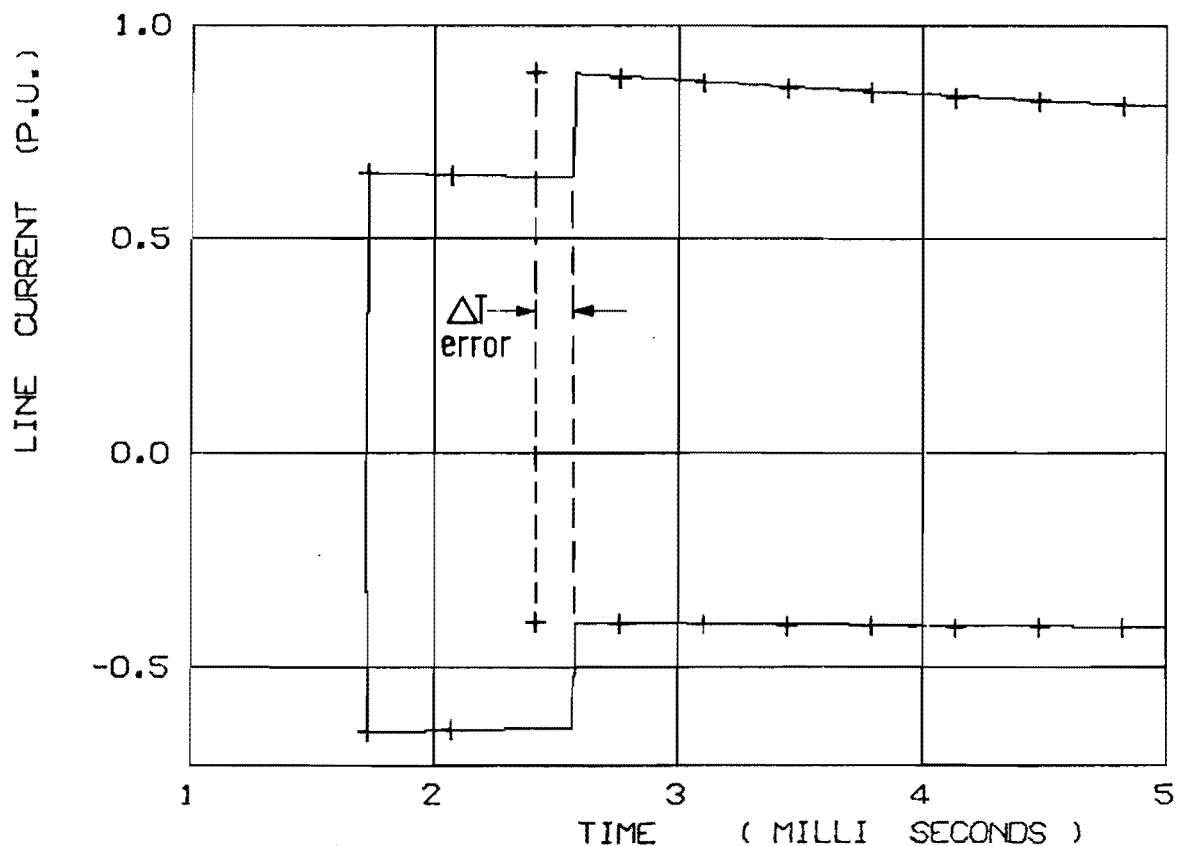
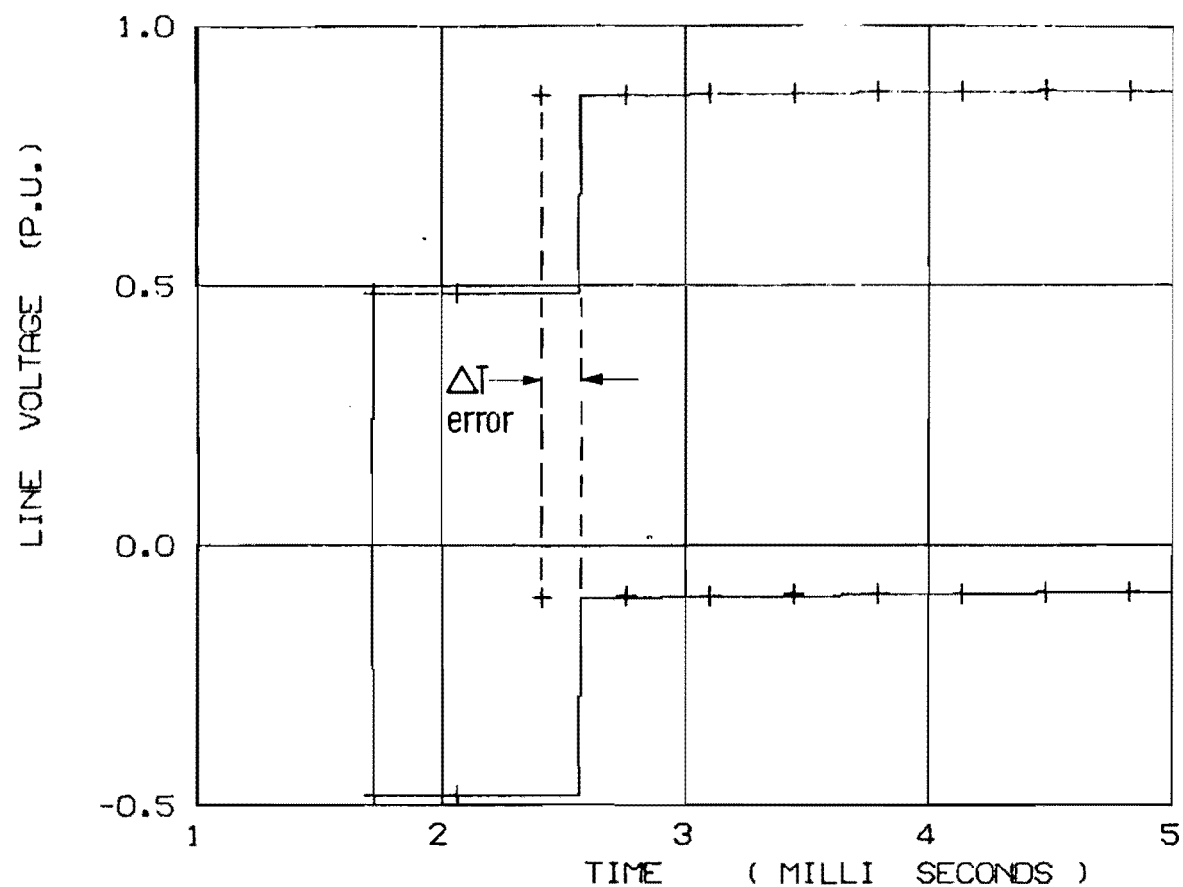


Fig. (11.18) Solution for a two circuit line using Point-Point M.O.C. and modal components. Solution at  $X = 500$  km. (Compare with Fig. (11.6).)



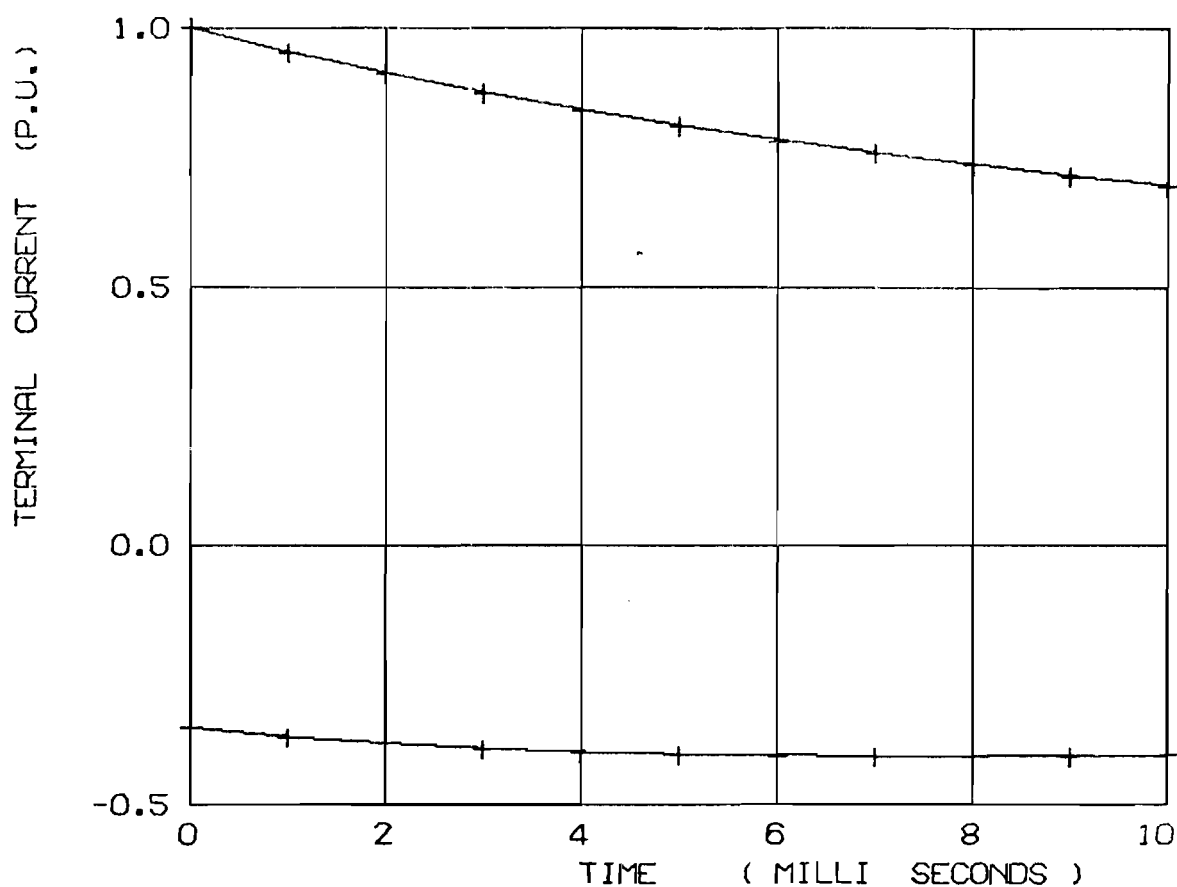


Fig. (11.19) Solution for two circuit line using Point-Point M.O.C. and modal components. (Compare with Figs (11.10) and (11.11).)

TIME SECONDS	I1 AT X=0 AMPS	I2 AT X=0 AMPS	V1 X=500KM VOLTS	V2 X=500KM VOLTS	I1 X=500KM AMPS	I2 X=500KM AMPS
0.0	2.8130E-03	-9.9198E-04	0.0	0.0	0.0	0.0
1.728E-04	2.8130E-03	-9.9198E-04	0.0	0.0	0.0	0.0
3.456E-04	2.8130E-03	-9.9198E-04	0.0	0.0	0.0	0.0
5.184E-04	2.8130E-03	-9.9198E-04	0.0	0.0	0.0	0.0
6.912E-04	2.7843E-03	-9.6324E-04	0.0	0.0	0.0	0.0
8.640E-04	2.7843E-03	-9.6324E-04	0.0	0.0	0.0	0.0
1.037E-03	2.6891E-03	-1.0585E-03	0.0	0.0	0.0	0.0
1.210E-03	2.6891E-03	-1.0585E-03	0.0	0.0	0.0	0.0
1.382E-03	2.6610E-03	-1.0304E-03	0.0	0.0	0.0	0.0
1.555E-03	2.6610E-03	-1.0304E-03	0.0	0.0	0.0	0.0
1.728E-03	2.6610E-03	-1.0304E-03	4.8126E-01	-4.8126E-01	1.8312E-03	-1.8312E-03
1.901E-03	2.6610E-03	-1.0304E-03	4.8126E-01	-4.8126E-01	1.8312E-03	-1.8312E-03
2.074E-03	2.5532E-03	-1.0832E-03	4.8126E-01	-4.8126E-01	1.8312E-03	-1.8312E-03
2.246E-03	2.5532E-03	-1.0832E-03	4.8126E-01	-4.8126E-01	1.8312E-03	-1.8312E-03
2.419E-03	2.5532E-03	-1.0832E-03	4.8140E-01	-4.8140E-01	1.8041E-03	-1.8041E-03
2.592E-03	2.5532E-03	-1.0832E-03	8.5750E-01	-1.0530E-01	2.4890E-03	-1.1192E-03
2.765E-03	2.5264E-03	-1.0564E-03	8.5750E-01	-1.0530E-01	2.4890E-03	-1.1192E-03
2.938E-03	2.5264E-03	-1.0564E-03	8.5750E-01	-1.0530E-01	2.4890E-03	-1.1192E-03
3.110E-03	2.4584E-03	-1.1244E-03	8.5764E-01	-1.0544E-01	2.4625E-03	-1.0927E-03
3.283E-03	2.4584E-03	-1.1244E-03	8.5764E-01	-1.0544E-01	2.4625E-03	-1.0927E-03
3.456E-03	2.4322E-03	-1.0981E-03	8.5764E-01	-1.0544E-01	2.4625E-03	-1.0927E-03
3.629E-03	2.4322E-03	-1.0981E-03	8.6340E-01	-9.9678E-02	2.4013E-03	-1.1538E-03
3.802E-03	2.4322E-03	-1.0981E-03	8.6354E-01	-9.9814E-02	2.3754E-03	-1.1279E-03
3.974E-03	2.4322E-03	-1.0981E-03	8.6354E-01	-9.9814E-02	2.3754E-03	-1.1279E-03
4.147E-03	2.3488E-03	-1.1302E-03	8.6354E-01	-9.9813E-02	2.3754E-03	-1.1279E-03
4.320E-03	2.3488E-03	-1.1302E-03	8.6354E-01	-9.9813E-02	2.3754E-03	-1.1279E-03
4.493E-03	2.3488E-03	-1.1302E-03	8.6367E-01	-9.9947E-02	2.3501E-03	-1.1025E-03
4.666E-03	2.3488E-03	-1.1302E-03	8.6886E-01	-9.4751E-02	2.2980E-03	-1.1546E-03
4.838E-03	2.3238E-03	-1.1052E-03	8.6886E-01	-9.4751E-02	2.2980E-03	-1.1546E-03
5.011E-03	2.3238E-03	-1.1052E-03	8.6886E-01	-9.4751E-02	2.2980E-03	-1.1546E-03
5.184E-03	2.2745E-03	-1.1544E-03	8.6900E-01	-9.4882E-02	2.2732E-03	-1.1298E-03

TABLE ( 11.1 )

DIRECT APPLICATION OF METHOD OF CHARACTERISTICS  
CONSTANT CURRENT FUNCTION APPROXIMATION

TIME SECONDS	I1 AT X=0 AMPS	I2 AT X=0 AMPS	V1 X=500KM VOLTS	V2 X=500KM VOLTS	I1 X=500KM AMPS	I2 X=500KM AMPS
0.0	2.8130E-03	-9.9198E-04	0.0	0.0	0.0	0.0
1.728E-04	2.8130E-03	-9.9198E-04	0.0	0.0	0.0	0.0
3.456E-04	2.8130E-03	-9.9198E-04	0.0	0.0	0.0	0.0
5.184E-04	2.8130E-03	-9.9198E-04	0.0	0.0	0.0	0.0
6.912E-04	2.7844E-03	-9.6335E-04	0.0	0.0	0.0	0.0
8.640E-04	2.7844E-03	-9.6335E-04	0.0	0.0	0.0	0.0
1.037E-03	2.6916E-03	-1.0561E-03	0.0	0.0	0.0	0.0
1.210E-03	2.6916E-03	-1.0561E-03	0.0	0.0	0.0	0.0
1.382E-03	2.6636E-03	-1.0282E-03	0.0	0.0	0.0	0.0
1.555E-03	2.6636E-03	-1.0282E-03	0.0	0.0	0.0	0.0
1.728E-03	2.6636E-03	-1.0282E-03	4.8133E-01	-4.8133E-01	1.8315E-03	-1.8315E-03
1.901E-03	2.6636E-03	-1.0282E-03	4.8133E-01	-4.8133E-01	1.8315E-03	-1.8315E-03
2.074E-03	2.5577E-03	-1.0794E-03	4.8133E-01	-4.8133E-01	1.8315E-03	-1.8315E-03
2.246E-03	2.5577E-03	-1.0794E-03	4.8133E-01	-4.8133E-01	1.8315E-03	-1.8315E-03
2.419E-03	2.5577E-03	-1.0794E-03	4.8147E-01	-4.8147E-01	1.8044E-03	-1.8044E-03
2.592E-03	2.5577E-03	-1.0794E-03	8.6057E-01	-1.0237E-01	2.4948E-03	-1.1141E-03
2.765E-03	2.5309E-03	-1.0527E-03	8.6057E-01	-1.0237E-01	2.4948E-03	-1.1141E-03
2.938E-03	2.5309E-03	-1.0527E-03	8.6071E-01	-1.0250E-01	2.4684E-03	-1.0876E-03
3.110E-03	2.4641E-03	-1.1195E-03	8.6071E-01	-1.0250E-01	2.4684E-03	-1.0876E-03
3.283E-03	2.4641E-03	-1.1195E-03	8.6071E-01	-1.0250E-01	2.4684E-03	-1.0876E-03
3.456E-03	2.4380E-03	-1.0934E-03	8.6620E-01	-9.7007E-02	2.4080E-03	-1.1480E-03
3.629E-03	2.4380E-03	-1.0934E-03	8.6634E-01	-9.7141E-02	2.3822E-03	-1.1222E-03
3.802E-03	2.4380E-03	-1.0934E-03	8.6634E-01	-9.7141E-02	2.3822E-03	-1.1222E-03
3.974E-03	2.4380E-03	-1.0934E-03	8.6634E-01	-9.7141E-02	2.3822E-03	-1.1222E-03
4.147E-03	2.3554E-03	-1.1248E-03	8.6634E-01	-9.7141E-02	2.3822E-03	-1.1222E-03
4.320E-03	2.3554E-03	-1.1248E-03	8.6634E-01	-9.7141E-02	2.3822E-03	-1.1222E-03
4.493E-03	2.3554E-03	-1.1248E-03	8.6647E-01	-9.7273E-02	2.3569E-03	-1.0969E-03
4.666E-03	2.3554E-03	-1.1248E-03	8.7144E-01	-9.2304E-02	2.3054E-03	-1.1485E-03
4.838E-03	2.3304E-03	-1.0999E-03	8.7144E-01	-9.2304E-02	2.3054E-03	-1.1485E-03
5.011E-03	2.3304E-03	-1.0999E-03	8.7144E-01	-9.2304E-02	2.3054E-03	-1.1485E-03
5.184E-03	2.2815E-03	-1.1487E-03	8.7157E-01	-9.2434E-02	2.2807E-03	-1.1238E-03

TABLE ( 11.2 )

DIRECT APPLICATION OF METHOD OF CHARACTERISTICS  
EXPONENTIAL CURRENT FUNCTION APPROXIMATION

TIME SECONDS	I1 AT X=0 AMPS	I2 AT X=0 AMPS	V1 X=500KM VOLTS	V2 X=500KM VOLTS	I1 X=500KM AMPS	I2 X=500KM AMPS
0.0	2.8130E-03	-9.9198E-04	0.0	0.0	0.0	0.0
1.728E-04	2.7891E-03	-1.0015E-03	0.0	0.0	0.0	0.0
3.456E-04	2.7657E-03	-1.0105E-03	0.0	0.0	0.0	0.0
5.184E-04	2.7428E-03	-1.0192E-03	0.0	0.0	0.0	0.0
6.912E-04	2.7203E-03	-1.0274E-03	0.0	0.0	0.0	0.0
8.640E-04	2.6983E-03	-1.0353E-03	0.0	0.0	0.0	0.0
1.037E-03	2.6768E-03	-1.0428E-03	0.0	0.0	0.0	0.0
1.210E-03	2.6556E-03	-1.0500E-03	0.0	0.0	0.0	0.0
1.382E-03	2.6349E-03	-1.0568E-03	0.0	0.0	0.0	0.0
1.555E-03	2.6146E-03	-1.0633E-03	0.0	0.0	0.0	0.0
1.728E-03	2.5947E-03	-1.0695E-03	4.8133E-01	-4.8133E-01	1.8315E-03	-1.8315E-03
1.901E-03	2.5752E-03	-1.0754E-03	4.8137E-01	-4.8137E-01	1.8247E-03	-1.8247E-03
2.074E-03	2.5561E-03	-1.0810E-03	4.8140E-01	-4.8140E-01	1.8179E-03	-1.8179E-03
2.246E-03	2.5373E-03	-1.0863E-03	4.8144E-01	-4.8144E-01	1.8111E-03	-1.8111E-03
2.419E-03	2.5189E-03	-1.0913E-03	4.8147E-01	-4.8147E-01	1.8044E-03	-1.8044E-03
2.592E-03	2.5008E-03	-1.0960E-03	8.6070E-01	-1.0230E-01	2.4870E-03	-1.1085E-03
2.765E-03	2.4831E-03	-1.1005E-03	8.6170E-01	-1.0137E-01	2.4695E-03	-1.1128E-03
2.938E-03	2.4657E-03	-1.1047E-03	8.6269E-01	-1.0046E-01	2.4523E-03	-1.1168E-03
3.110E-03	2.4486E-03	-1.1087E-03	8.6365E-01	-9.9558E-02	2.4355E-03	-1.1205E-03
3.283E-03	2.4318E-03	-1.1124E-03	8.6460E-01	-9.8675E-02	2.4189E-03	-1.1241E-03
3.456E-03	2.4153E-03	-1.1160E-03	8.6554E-01	-9.7807E-02	2.4026E-03	-1.1274E-03
3.629E-03	2.3991E-03	-1.1192E-03	8.6646E-01	-9.6954E-02	2.3866E-03	-1.1305E-03
3.802E-03	2.3832E-03	-1.1223E-03	8.6736E-01	-9.6116E-02	2.3709E-03	-1.1334E-03
3.974E-03	2.3676E-03	-1.1252E-03	8.6825E-01	-9.5291E-02	2.3555E-03	-1.1361E-03
4.147E-03	2.3523E-03	-1.1279E-03	8.6913E-01	-9.4481E-02	2.3403E-03	-1.1386E-03
4.320E-03	2.3372E-03	-1.1304E-03	8.6999E-01	-9.3685E-02	2.3254E-03	-1.1409E-03
4.493E-03	2.3223E-03	-1.1327E-03	8.7084E-01	-9.2902E-02	2.3108E-03	-1.1430E-03
4.666E-03	2.3077E-03	-1.1348E-03	8.7167E-01	-9.2132E-02	2.2964E-03	-1.1450E-03
4.838E-03	2.2934E-03	-1.1368E-03	8.7249E-01	-9.1375E-02	2.2822E-03	-1.1467E-03
5.011E-03	2.2793E-03	-1.1385E-03	8.7330E-01	-9.0630E-02	2.2683E-03	-1.1483E-03
5.184E-03	2.2654E-03	-1.1402E-03	8.7410E-01	-8.9898E-02	2.2546E-03	-1.1498E-03

TABLE ( 11.3 )

EXACT SOLUTION. COMPONENTS CALCULATED USING  
CARSON'S SOLUTION.

TIME SECONDS	I1 AT X=0 AMPS	I2 AT X=0 AMPS	V1 X=500KM VOLTS	V2 X=500KM VOLTS	I1 X=500KM AMPS	I2 X=500KM AMPS
0.0	2.8130E-03	-9.9198E-04	0.0	0.0	0.0	0.0
1.728E-04	2.7891E-03	-1.0015E-03	0.0	0.0	0.0	0.0
3.456E-04	2.7657E-03	-1.0105E-03	0.0	0.0	0.0	0.0
5.184E-04	2.7428E-03	-1.0192E-03	0.0	0.0	0.0	0.0
6.912E-04	2.7203E-03	-1.0275E-03	0.0	0.0	0.0	0.0
8.640E-04	2.6983E-03	-1.0353E-03	0.0	0.0	0.0	0.0
1.037E-03	2.6767E-03	-1.0429E-03	0.0	0.0	0.0	0.0
1.210E-03	2.6556E-03	-1.0501E-03	0.0	0.0	0.0	0.0
1.382E-03	2.6349E-03	-1.0569E-03	0.0	0.0	0.0	0.0
1.555E-03	2.6146E-03	-1.0634E-03	0.0	0.0	0.0	0.0
1.728E-03	2.5947E-03	-1.0696E-03	4.81133E-01	-4.81133E-01	1.8315E-03	-1.8315E-03
1.901E-03	2.5752E-03	-1.0755E-03	4.81137E-01	-4.81137E-01	1.8247E-03	-1.8247E-03
2.074E-03	2.5560E-03	-1.0810E-03	4.81140E-01	-4.81140E-01	1.8179E-03	-1.8179E-03
2.246E-03	2.5372E-03	-1.0863E-03	4.81144E-01	-4.81144E-01	1.8111E-03	-1.8111E-03
2.419E-03	2.5188E-03	-1.0913E-03	4.81147E-01	-4.81147E-01	1.8044E-03	-1.8044E-03
2.592E-03	2.5007E-03	-1.0961E-03	8.6071E-01	-1.0230E-01	2.4870E-03	-1.1085E-03
2.765E-03	2.4830E-03	-1.1006E-03	8.6171E-01	-1.0137E-01	2.4695E-03	-1.1128E-03
2.938E-03	2.4656E-03	-1.1048E-03	8.6269E-01	-1.0046E-01	2.4523E-03	-1.1169E-03
3.110E-03	2.4485E-03	-1.1088E-03	8.6366E-01	-9.9555E-02	2.4355E-03	-1.1205E-03
3.283E-03	2.4317E-03	-1.1125E-03	8.6461E-01	-9.8671E-02	2.4189E-03	-1.1241E-03
3.456E-03	2.4152E-03	-1.1160E-03	8.6555E-01	-9.7801E-02	2.4026E-03	-1.1274E-03
3.629E-03	2.3990E-03	-1.1193E-03	8.6647E-01	-9.6947E-02	2.3866E-03	-1.1305E-03
3.802E-03	2.3831E-03	-1.1224E-03	8.6738E-01	-9.6108E-02	2.3709E-03	-1.1334E-03
3.974E-03	2.3675E-03	-1.1253E-03	8.6827E-01	-9.5282E-02	2.3555E-03	-1.1361E-03
4.147E-03	2.3521E-03	-1.1280E-03	8.6915E-01	-9.4471E-02	2.3403E-03	-1.1386E-03
4.320E-03	2.3370E-03	-1.1305E-03	8.7001E-01	-9.3674E-02	2.3254E-03	-1.1409E-03
4.493E-03	2.3222E-03	-1.1328E-03	8.7086E-01	-9.2890E-02	2.3108E-03	-1.1430E-03
4.666E-03	2.3076E-03	-1.1349E-03	8.7170E-01	-9.2119E-02	2.2964E-03	-1.1450E-03
4.838E-03	2.2933E-03	-1.1369E-03	8.7252E-01	-9.1362E-02	2.2822E-03	-1.1467E-03
5.011E-03	2.2792E-03	-1.1386E-03	8.7333E-01	-9.0616E-02	2.2683E-03	-1.1483E-03
5.184E-03	2.2653E-03	-1.1403E-03	8.7413E-01	-8.9883E-02	2.2545E-03	-1.1498E-03

TABLE ( 11.4 )

LINE POINT TO LINE POINT METHOD OF CHARACTERISTICS  
AND MODAL COMPONENTS.

## SECTION 12

LINE POINT TO LINE POINT FORMULATION OF THE METHOD  
OF CHARACTERISTICS FOR THE LOSSY T.L.E.s

A formulation of this form will alleviate the problems of matching the solution grid with the characteristic grid in the  $(x,t)$  plane. This will be particularly helpful when the method of characteristics is used in conjunction with modal components for multiple circuit transmission lines. (sections 11.2.3), 11.2.4)).

A line point to line point solution could have been obtained by eliminating line intermediate solution points (section 10)) in terms of the terminal solution points. Little would have been gained. The resulting solution would still be approximate in terms of variable function approximations with propagation errors dependent on  $\Delta T$ . Alternatively a solution of the same form can be fabricated using the transmission line's impulse responses. This leads to a solution of an entirely different nature. Whereas the former is derived in terms of discrete approximations to line propagation, the latter is continuous in distance and no such approximations are needed. Errors in the second method derive from a stepwise approximation to the terminal signals. In the power systems transient context this is no great disadvantage as terminal signals are often stepwise in nature.

## 12.1) DERIVATION OF CHARACTERISTIC EQUATIONS

Here the transmission line responses will be written initially Laplace transformed with respect to time. The inverse transforms will then be taken to give time domain equations.

Consider a point  $x \geq 0$  on a semi-infinite transmission line. The applied voltage at end  $x = 0$  gives rise at point  $x$  to the signals:

$$\begin{aligned} V(x,s) &= F(x,s) V_0(s) \\ I(x,s) &= G(x,s) V_0(s) \end{aligned} \tag{12.1}$$

Alternatively equations (12.1) could have been written in terms of line input current:

$$\begin{aligned} V(x,s) &= H(x,s) I_0(s) \\ I(x,s) &= F(x,s) I_0(s) \end{aligned} \tag{12.2}$$

where  $F(x,s)$ ,  $G(x,s)$  and  $H(x,s)$  are Laplace transforms of the transmission line's impulse responses (APPENDIX (12.1)).

Adding equations (12.1) and (12.2) gives:

$$\begin{aligned} 2V(x,s) &= F(x,s)V_0(s) + H(x,s)I_0(s) \\ 2I(x,s) &= G(x,s)V_0(s) + F(x,s)I_0(s) \end{aligned} \tag{12.3}$$

Similarly, for a semi-infinite line extending from  $x = -\infty$  to  $x = d$  we can write for  $x \leq d$ :

$$\begin{aligned} 2V(x,s) &= F(d-x,s)V_d(s) - H(d-x,s)I_d(s) \\ -2I(x,s) &= G(d-x,s)V_d(s) - F(d-x,s)I_d(s) \end{aligned} \tag{12.4}$$

Consider now a finite line from  $x = 0$  to  $x = d$ . Combining forward and backward propagating signals from ends  $x = 0$  and  $x = d$  respectively at point  $0 < x < d$  gives for the functions  $(V + Z_C I)$  and  $(V - Z_C I)$ :

$$2(V + Z_C I)_x = B1_x \cdot V_0 + B2_x \cdot I_0 + B3_{(d-x)} \cdot V_d - B4_{(d-x)} \cdot I_d \quad (12.5)$$

$$2(V - Z_C I)_x = B3_x \cdot V_0 + B4_x \cdot I_0 + B1_{(d-x)} \cdot V_d - B2_{(d-x)} \cdot I_d$$

$$\text{where } B1 = F + Z_C G$$

$$B2 = H + Z_C F$$

$$B3 = F - Z_C G$$

$$B4 = H - Z_C F$$

Taking the inverse Laplace transforms of (12.5) gives:

$$\begin{aligned} 2(v + Z_C i)_x(t) = & \int_{\frac{x}{a}}^t b1(x,u) v_0(t-u) du + \int_{\frac{x}{a}}^t b2(x,u) i_0(t-u) du \\ & + \int_{\frac{d-x}{a}}^t b3(d-x,u) v_d(t-u) du - \int_{\frac{d-x}{a}}^t b4(d-x,u) i_d(t-u) du \quad (12.6) \end{aligned}$$

$$\begin{aligned} 2(v - Z_C i)_x = & \int_{\frac{x}{a}}^t b3(x,u) v_0(t-u) du + \int_{\frac{x}{a}}^t b4(x,u) i_0(t-u) du \\ & + \int_{\frac{d-x}{a}}^t b1(d-x,u) v_d(t-u) du - \int_{\frac{d-x}{a}}^t b2(d-x,u) i_d(t-u) du \quad (12.7) \end{aligned}$$

where: (APPENDIX (12.1))



$$b1(x,t) = e^{-\rho t} \left\{ 2\delta\left(t - \frac{x}{a}\right) - \sigma \left[ I_0(\sigma p) - \left(t + \frac{x}{a}\right) \frac{I_1(\sigma p)}{p} \right] \right\}$$

$$b2(x,t) = z_c e^{-\rho t} \left\{ 2\delta\left(t - \frac{x}{a}\right) + \sigma \left[ I_0(\sigma p) + \left(t + \frac{x}{a}\right) \frac{I_1(\sigma p)}{p} \right] \right\}$$

$$b3(x,t) = \sigma e^{-\rho t} \left\{ I_0(\sigma p) - \left(t - \frac{x}{a}\right) \frac{I_1(\sigma p)}{p} \right\}$$

$$b4(x,t) = z_c \sigma e^{-\rho t} \left\{ I_0(\sigma p) + \left(t - \frac{x}{a}\right) \frac{I_1(\sigma p)}{p} \right\}$$

all of which apply for  $t \geq \frac{x}{a}$  and where  $\rho = \frac{1}{2} \left( \frac{R}{L} + \frac{G}{C} \right)$ ,

$\sigma = \frac{1}{2} \left( \frac{R}{L} - \frac{G}{C} \right)$ ,  $z_c = \sqrt{\frac{L}{C}}$ ,  $a = 1/\sqrt{LC}$ ,  $p = \sqrt{t^2 - \left(\frac{x}{a}\right)^2}$ ,  $\delta$  is a

dirac delta function and  $I_0$  and  $I_1$  are modified Bessel functions of the first kind.

Integrating out the delta functions, equations (12.6) and (12.7) can be rewritten:

$$\begin{aligned} (v + z_c i)(x,t) &= (v + z_c i)_{(0, t - \frac{x}{a})} e^{-\frac{\rho x}{a}} \\ &+ \int_{\frac{x}{a}}^t W1(x,u) v_0(t-u) du + z_c \int_{\frac{x}{a}}^t W2(x,u) i_0(t-u) du \\ &+ \int_{\frac{d-x}{a}}^t W3(d-x,u) v_d(t-u) du - z_c \int_{\frac{d-x}{a}}^t W4(d-x,u) i_d(t-u) du \end{aligned} \quad (12.8)$$

$$\begin{aligned}
(v - Z_C i)(x, t) &= (v - Z_C i)\left(d, t - \frac{d-x}{a}\right) e^{-\frac{\rho(d-x)}{a}} \\
&+ \int_{\frac{x}{a}}^t W3(x, u) v_0(t-u) du + Z_C \int_{\frac{x}{a}}^t W4(x, u) i_0(t-u) du \\
&+ \int_{\frac{d-x}{a}}^t W1(d-x, u) v_d(t-u) du - Z_C \int_{\frac{d-x}{a}}^t W2(d-x, u) i_d(t-u) du
\end{aligned} \tag{12.9}$$

where the weighting functions are redefined as:

$$\begin{aligned}
W1(x, t) &= \frac{\sigma}{2} e^{-\rho t} \left\{ \left(t + \frac{x}{a}\right) \frac{I_1(\sigma p)}{p} - I_0(\sigma p) \right\} \\
W2(x, t) &= \frac{\sigma}{2} e^{-\rho t} \left\{ \left(t + \frac{x}{a}\right) \frac{I_1(\sigma p)}{p} + I_0(\sigma p) \right\} \\
W3(x, t) &= \frac{\sigma}{2} e^{-\rho t} \left\{ I_0(\sigma p) - \left(t - \frac{x}{a}\right) \frac{I_1(\sigma p)}{p} \right\} \\
W4(x, t) &= \frac{\sigma}{2} e^{-\rho t} \left\{ I_0(\sigma p) + \left(t - \frac{x}{a}\right) \frac{I_1(\sigma p)}{p} \right\}
\end{aligned}$$

In equations (12.8) and (12.9) the distorting and non-distorting components of propagation have been separated. The distorting terms are contained in the integrals. For dispersionless transmission  $\sigma = 0$  and the integral terms are zero. The equations then reduce to those derived in section 7.4.2).

## 12.2) SOLUTION OF CHARACTERISTIC EQUATIONS

By setting  $x = d$  in equation (12.8) and  $x = 0$  in equation (12.9) the equations can be used to solve directly between the transmission line terminals. The weighting functions for such a solution can be seen in Fig. (12.3).

By setting  $v_d$  and  $i_d$  identically to zero equations (12.8) and (12.9) can be used to solve for propagation on a semi-infinite transmission line. Such a solution is carried out in section 12.2.4).

For dispersionless transmission these equations can be used to obtain solutions to transmission problems as in sections 4.4.3), 4.4.4), 7.4.1), 7.4.2) and 9.2). However, for transmission where distortion is present the integral terms are non-zero and must be evaluated.

#### 12.2.1) Evaluation of Convolution Integrals

Evaluation of the integrals in equations (12.8) and (12.9) requires the line variables to be known functionally in time. In general they are not so known and it is necessary to find discrete approximations to the integrals. The simplest mathematical discretization assumes the line variables to be constant over the range of integration giving a stepwise function in time. This is also a realistic approximation for switching and surge propagation problems and will be used here. When this is done the integral terms take the form:

$$\int_{\frac{x}{a}}^t W(x,u) f(t-u) du \approx \sum_{k=N1}^N W(x,k\Delta T) \cdot f(t-k\Delta T) \quad (12.10)$$

$$\text{where } (N1 - 1)\Delta T = \frac{x}{a}, \quad N \cdot \Delta T = t \quad \text{and} \quad W(x,k\Delta T) = \int_{(k-1)\Delta T}^{k\Delta T} W(x,u) du$$

The problem is reduced to that of integrating the weighting function.

For computation over a long period of time the upper limit of integration can be reduced to include only the range of values for which the weighting function is sensibly non-zero. The effects of limiting the range of integration are illustrated in section 12.2.3).

### 12.2.2) Integration of Weighting Functions

The integrals of the weighting functions can be evaluated numerically or by using series and asymptotic expansions. Typical weighting functions are shown in Figs (12.3) and (12.4). It will be seen that the function rate of change diminishes rapidly with time. For these kinds of function numerical integration by the mid-ordinate rule is particularly suitable. For the tails of the weighting functions or for solutions involving small time increments sufficient accuracy will often be obtained using only one application of the rule. For large time increments or where the function has high rates of change the range of integration can be subdivided and the error expressed in terms of the first derivative at each end of the undivided integration range. (An important advantage in this case as the first derivatives of the weighting functions can be found analytically.) The mid-ordinate rule of integration is expressed:

$$\int_x^{x+\Delta x} f(x) dx = f\left(x + \frac{\Delta x}{2}\right) \cdot \Delta x \quad (12.11)$$

For higher accuracy the range of integration can be subdivided and equation (12.11) applied to each subdivision. The integration truncation error of such a procedure is given by:

$$\text{Truncation error} = \frac{h}{c} (f'_n - f'_0) \quad (12.12)$$

where  $f'_n$  and  $f'_0$  are the function first derivatives at each end of the integration range and  $h$  is the width of each subdivision (REDISH, 1961).

### 12.2.3) Example Solutions: Unit Step of Voltage Impressed on a Semi-Infinite Series Resistance Transmission Line

To solve this problem, equations (12.8) and (12.9) are used with  $v_d$  and  $i_d$  set to zero. They then reduce to:

$$\begin{aligned} (v + Z_c i)(x, t) &= (v + Z_c i)\left(0, t - \frac{x}{a}\right) e^{-\rho \frac{x}{a}} \\ &+ \int_{\frac{x}{a}}^t W1(x, u) v_0(t-u) du + Z_c \int_{\frac{x}{a}}^t W2(x, u) i_0(t-u) du \end{aligned} \quad (12.13)$$

$$\begin{aligned} (v - Z_c i)(x, t) &= \int_{\frac{x}{a}}^t W3(x, u) v_0(t-u) du \\ &+ Z_c \int_{\frac{x}{a}}^t W4(x, u) i_0(t-u) du \end{aligned} \quad (12.14)$$

To obtain the line terminal current,  $x$  is set to zero in equation (12.14). The transmission line considered is the same as that used in section 10) with a surge impedance of 1 p.u. The transmission line responses were calculated at the line terminal and at a distance of  $x = 1$  p.u. from the terminal. The weighting functions used are plotted in Figs (12.3) and (12.4). Throughout these examples the series resistance per unit system was used. The application of the per unit system to the convolution integrals is given in

APPENDIX (12.3) together with a list of weighting functions and their first derivatives.

(a) Solution Increments

Example (1): Here a solution time increment of  $\Delta T = 1 \times 10^{-2}$  p.u. was used. The weighting functions were evaluated using a single application of the mid-ordinate rule over each range of integration. The resulting solution followed CARSON'S (1926) solution closely as seen in tables (12.1) and (12.2). It is also shown as the continuous curve in Figs (12.1) and (12.2).

Example (2): This solution is identical to that of example (1) except that the time increment used was 0.5 p.u. The solution is given in table (12.3) and is shown as '+' in Fig. (12.1).

Example (3): Here the weighting functions were integrated with a maximum error magnitude of  $1 \times 10^{-5}$ . In examples (1) and (2) only a single application of the mid-ordinate rule was used for each integration. Apart from this, example (3) was identical to example (2). The solution is given in table (12.4) and is shown as '+' in Fig. (12.2).

Comment on examples (2) and (3)

To obtain an error magnitude of  $\leq 1 \times 10^{-5}$  for the integral

$$\int_0^{0.5} W_3(0,t) dt$$

in example (3) it was necessary to use 43 subdivisions of the integral range; i.e. the function  $W_3(0,t)$  had to be evaluated 43 times. For the same integral in example (2) no

subdivision was used and  $W3(0,t)$  evaluated only once. In this case, the error magnitude was 0.018. While this is the most pronounced example of the extra computation required, at no stage in example (3) did the number of subdivisions required reduce to 1. On comparing Figs (12.1) and (12.2) it is seen that no real improvement was effected by this extra effort. From this it can be observed that the solution is more sensitive to changes in the solution time increment than to changes in the accuracy of integrating the weighting function. This observation can save considerable computing time. In both examples the toe values of the wave are exact after which the terminal currents are low, the line voltages are high and the line currents are followed fairly closely.

(b) Truncating the Convolution Integrals

In the next three examples the effect of truncating the convolution integrals will be investigated. In all cases a solution time increment of 0.1 p.u. was used and the integrals of the weighting functions were evaluated with a maximum error magnitude of  $1 \times 10^{-5}$ .

Example (4): In this example the full range of integration was used. This solution is shown as the continuous curve in Figs (12.5) and (12.6).

Example (5): Here the range of all the convolution integrals was limited to a maximum of 10 p.u. This is equivalent to assuming that all the weighting functions are zero after time equals 10 p.u. From Figs (12.3) and (12.4) it is seen that while this may include the bulk of the integral for functions  $W1$  and  $W3$  it yields a poor approximation to the integrals for functions  $W2$  and  $W4$ . As would be expected, the result

departs markedly from the unlimited solution as is shown in Figs (12.5) and (12.6).

Example (6): Here a selective limiting of the integrals is undertaken with functions W1 and W3 being assumed zero after 10 p.u. time but functions W2 and W4 remaining unlimited. The results, given in Figs (12.5) and (12.6), show a marked improvement over the results obtained in example (5). For the terminal current this could possibly be tolerated as a solution of sufficient accuracy. However, for the line solution a greater range of integration would be required to achieve a similarly tolerable solution.

Comment on examples (5) and (6)

For solutions involving long time spans or many solution points, truncation of the convolution integrals can be a satisfactory means of reducing computing effort without compromising the required solution accuracy. Weighting functions W2 and W4 are always more slowly decaying than W1 and W3 enabling truncation to be used on W1 and W3 before W2 and W4. As illustrated in Fig. (12.5) such limits need not significantly reduce solution accuracy.



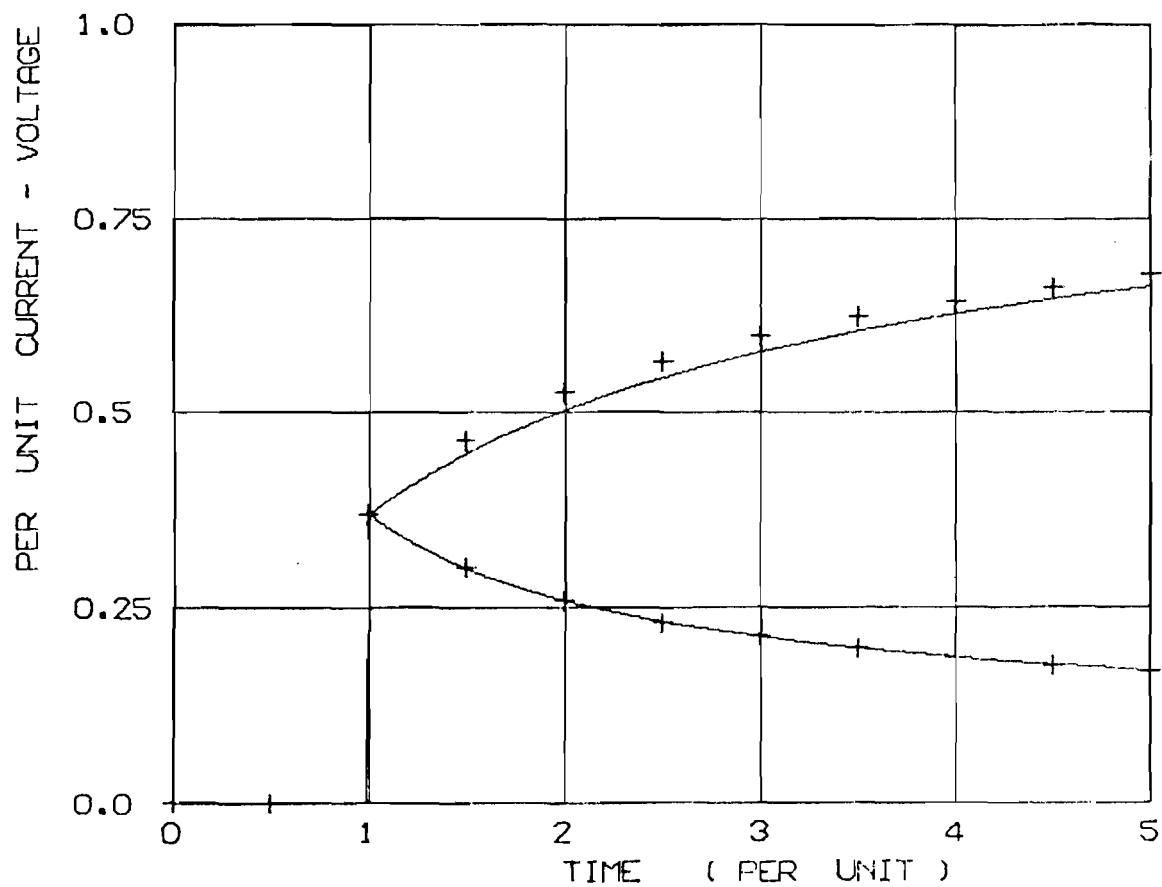
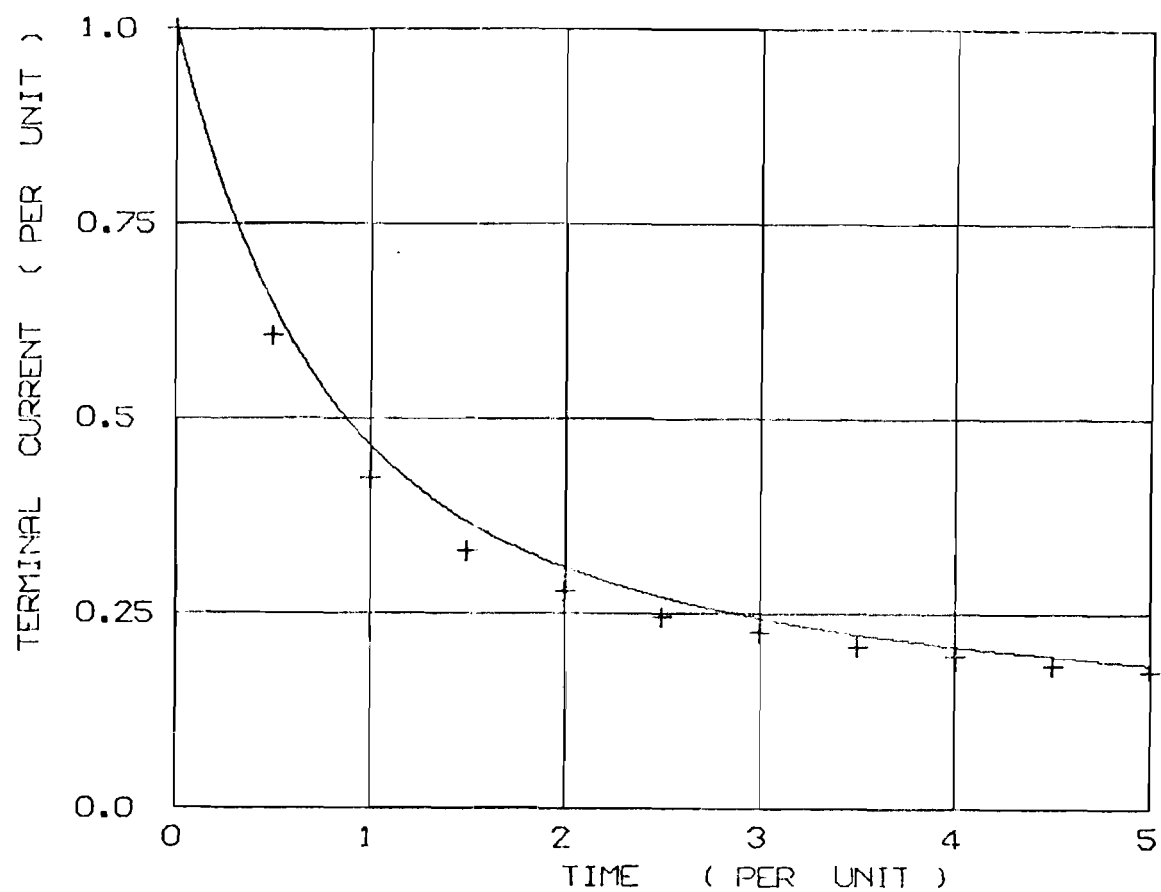


Fig. (12.1) Solution examples (1) and (2).

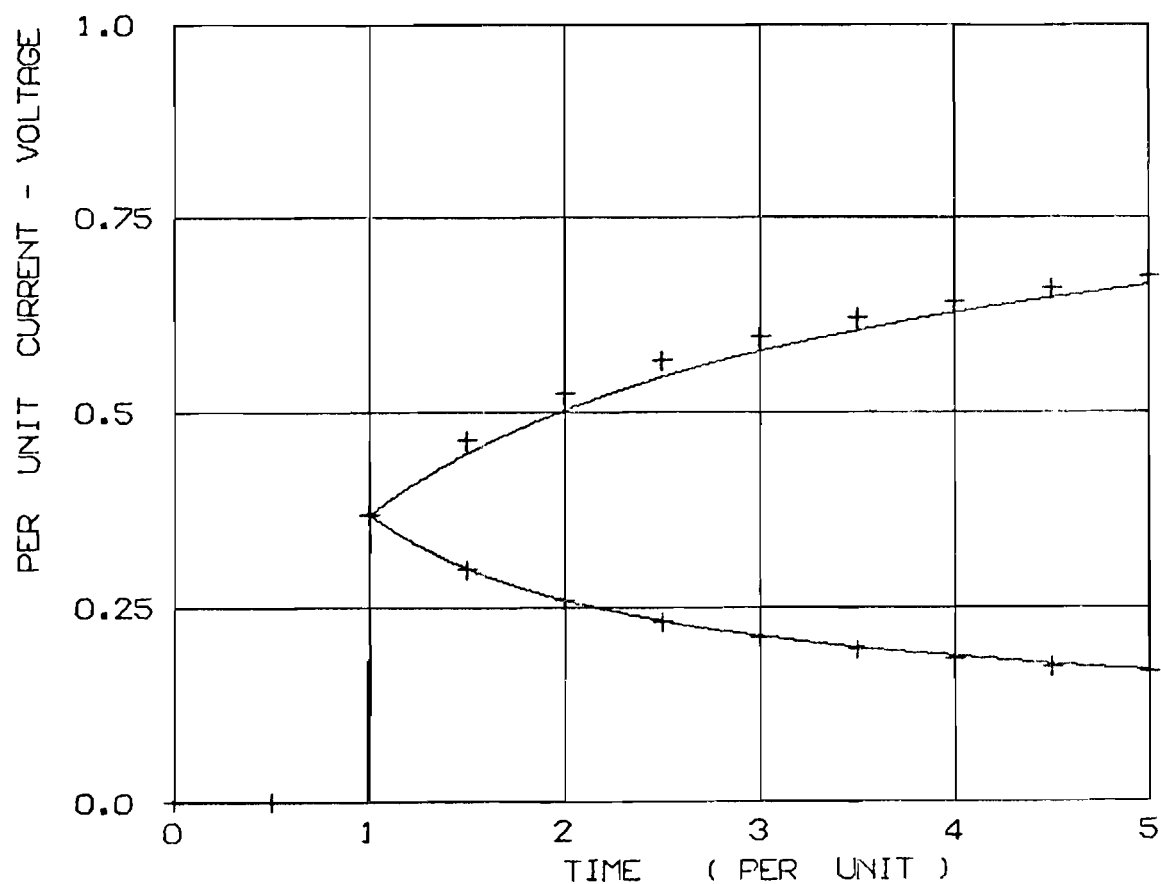
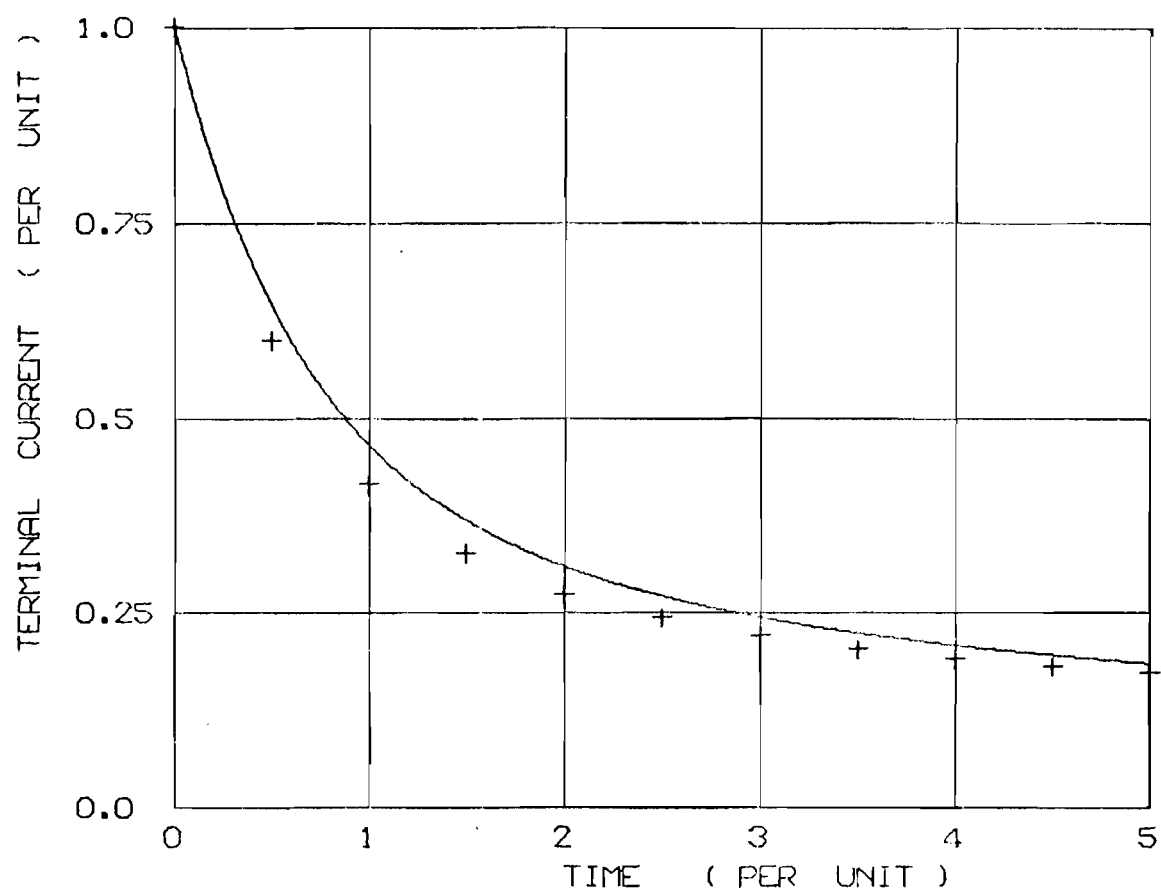


Fig. (12.2) Solution examples (1) and (3).

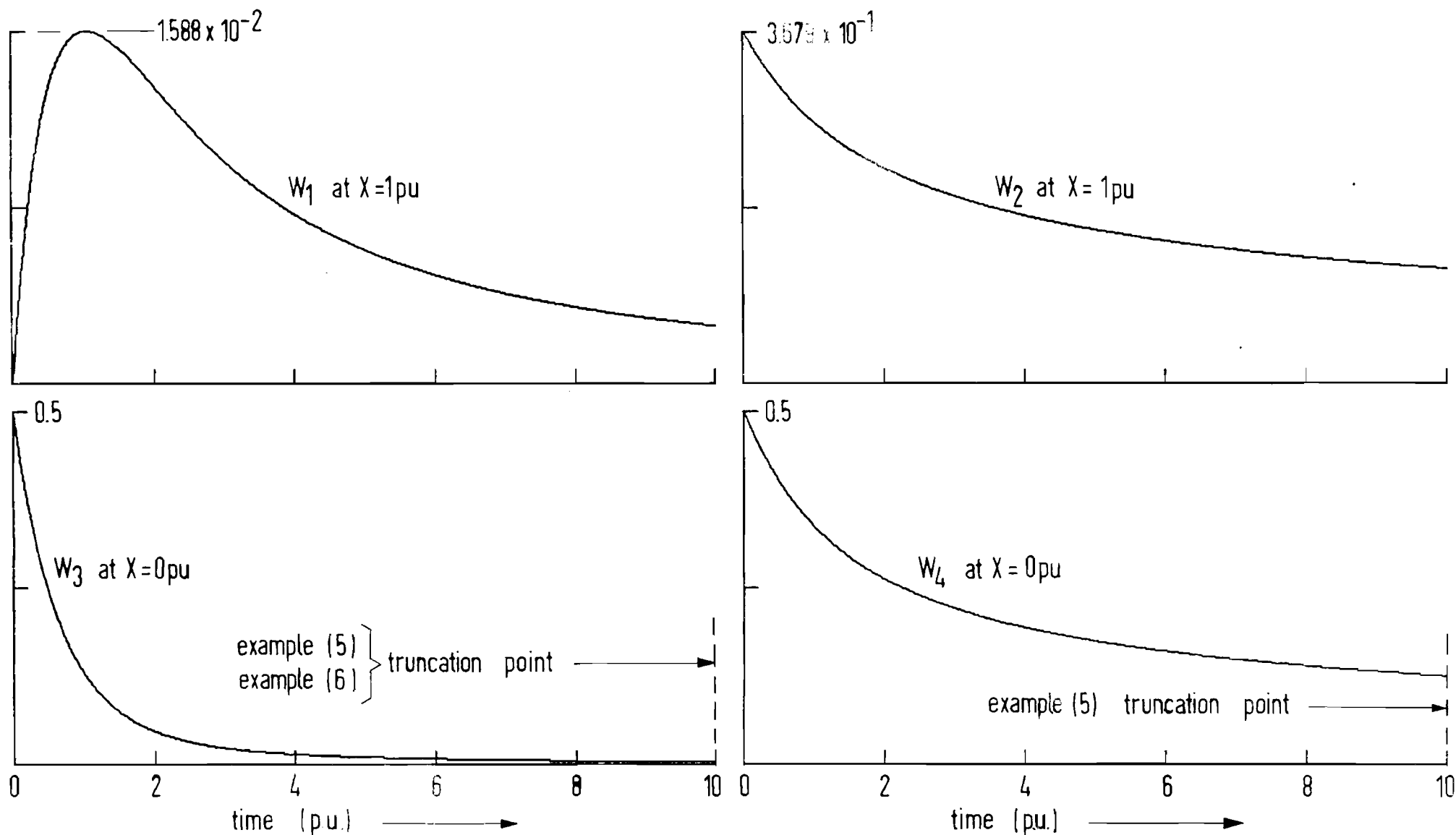


Fig. (12.3) Solution weighting functions for series resistance line.  $W_3$  and  $W_4$  as used for line terminal solution.

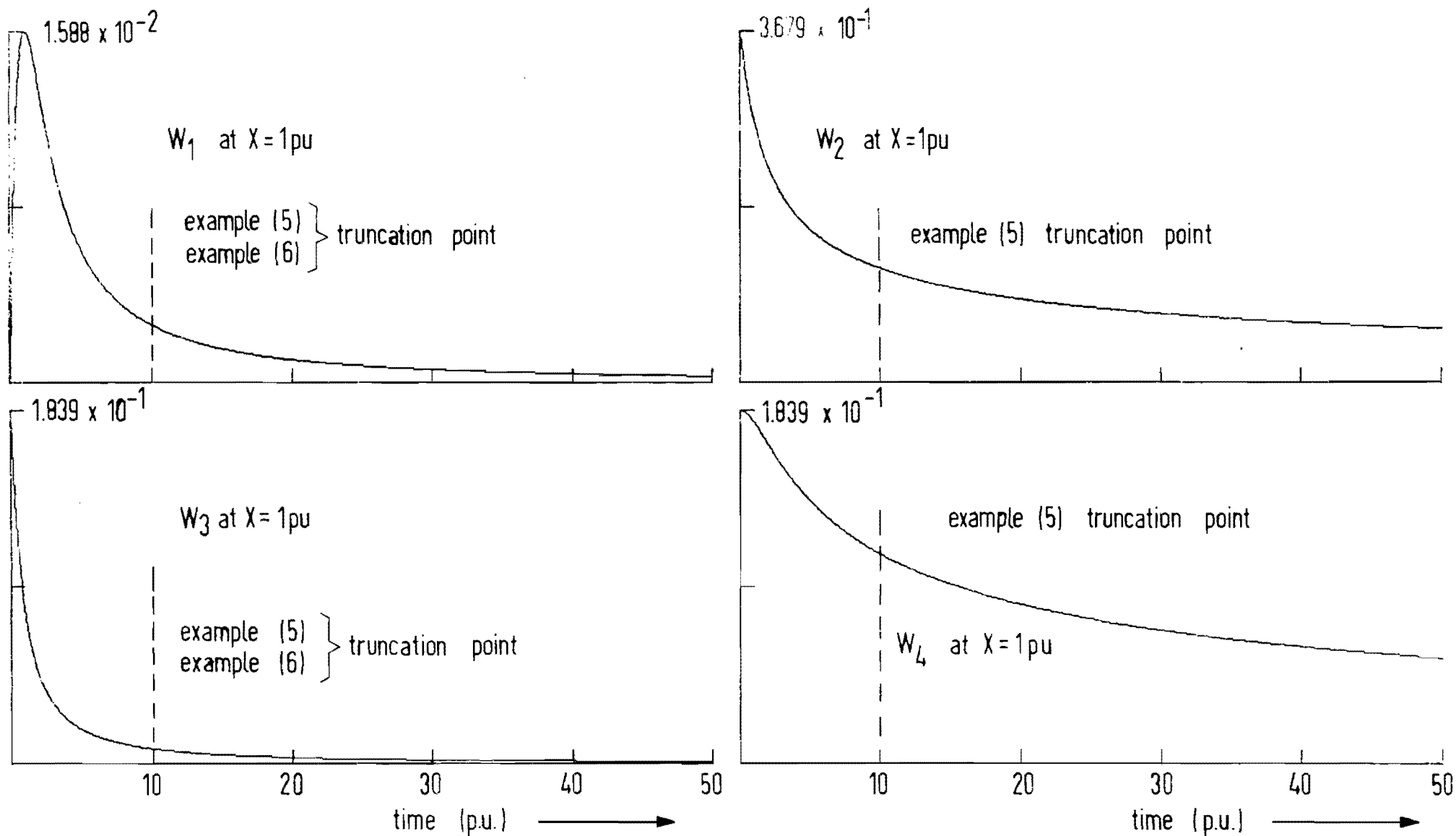


Fig. (12.4) Solution weighting functions for series resistance line - as used for solution at  $X=1$  p.u. 194

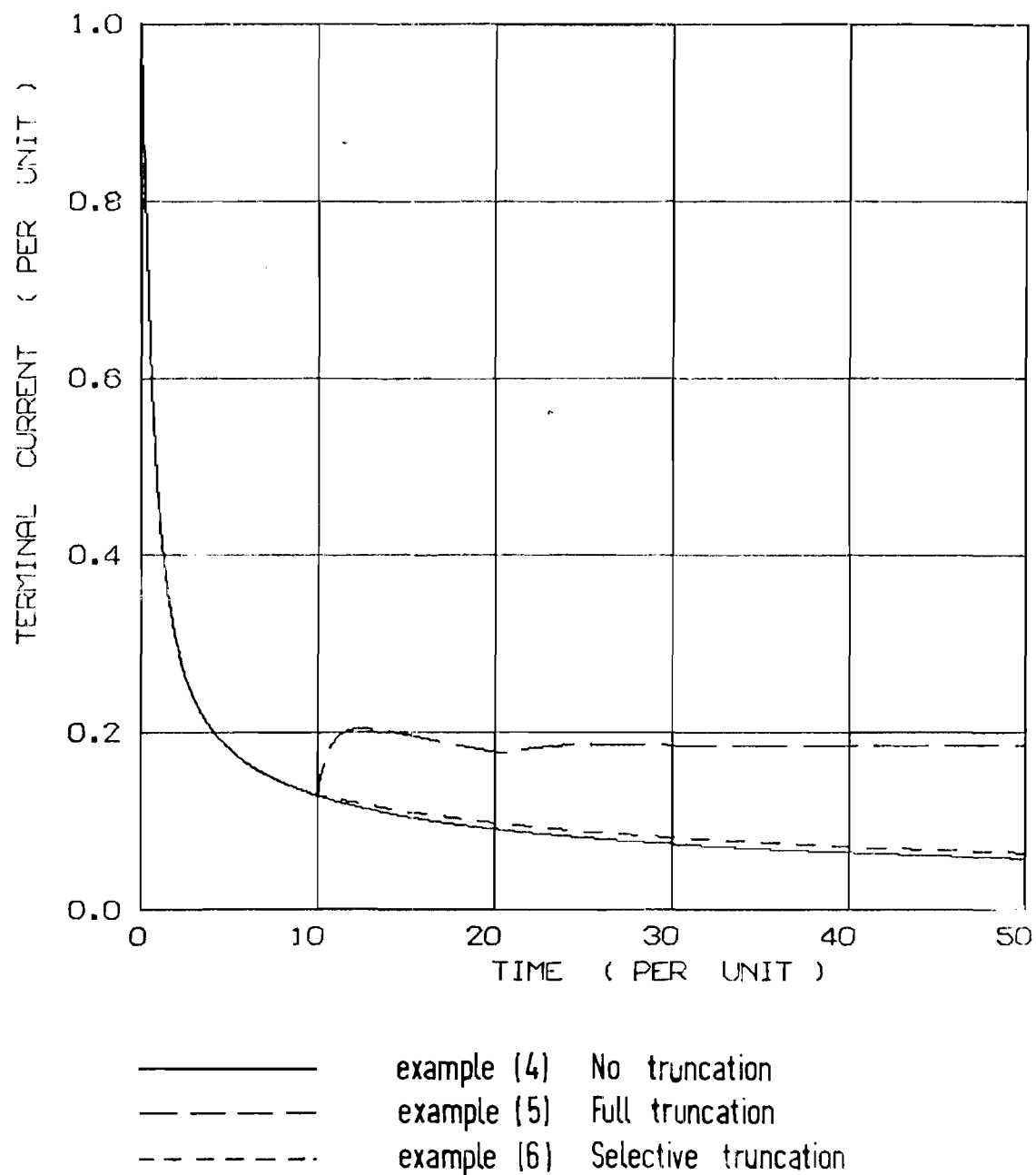


Fig. (12.5) Solution examples (4), (5) and (6). The effect of truncating the convolution integrals.

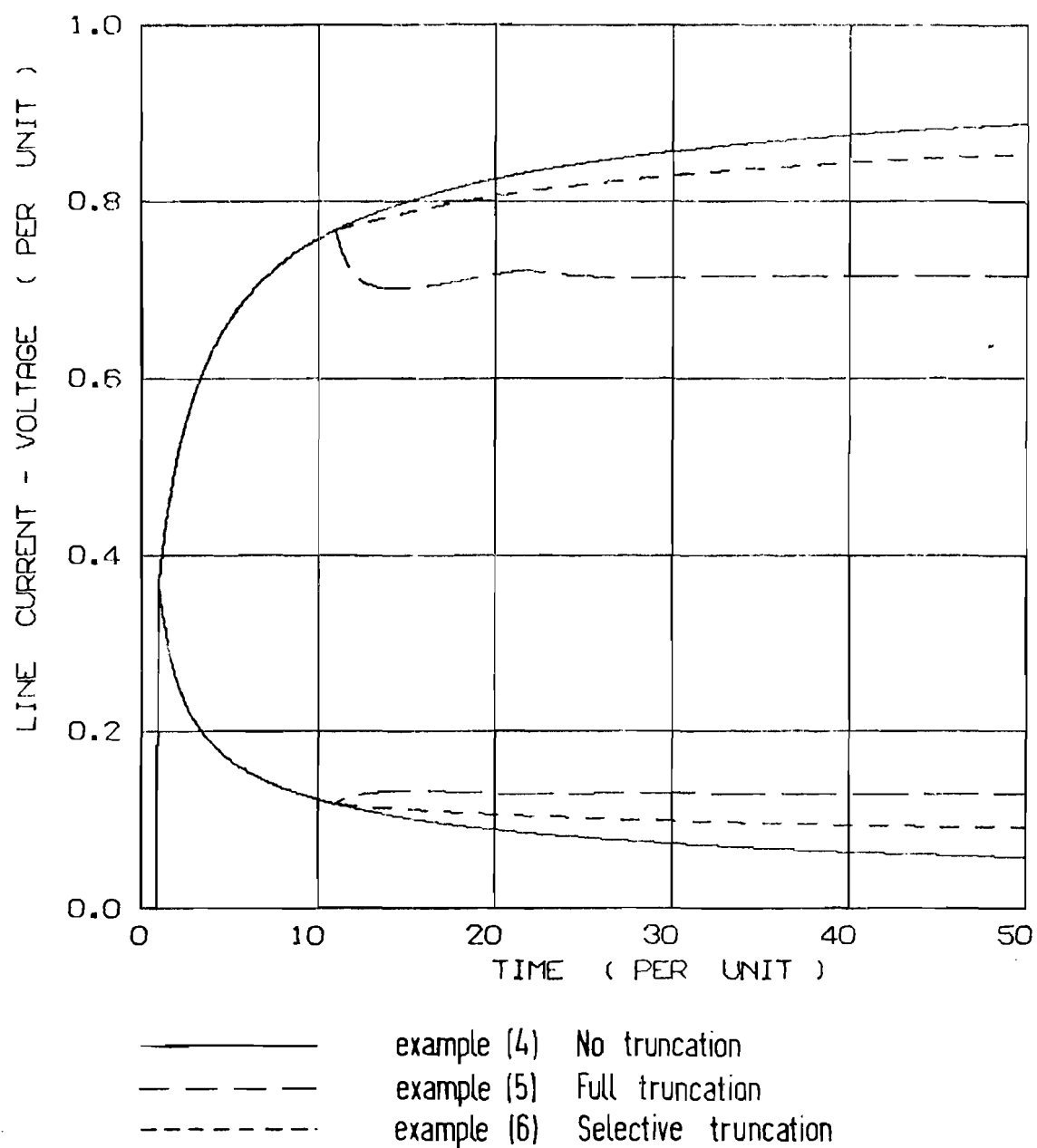


Fig. (12.6) Solution examples (4), (5) and (6). The effect of truncating the convolution integrals. Solution at  $X = 1$  p.u.

TIME (P.U.)	X = 0 (P.U.) CURRENT (P.U.)	X = 1 (P.U.)	
		VOLTAGE (P.U.)	CURRENT (P.U.)
0.0	1.00000	0.0	0.0
0.5	0.64503	0.0	0.0
1.0	0.46576	0.36788	0.36788
1.5	0.36744	0.44522	0.29850
2.0	0.30851	0.50118	0.25754
2.5	0.27005	0.54356	0.23078
3.0	0.24301	0.57688	0.21172
3.5	0.22281	0.60389	0.19719
4.0	0.20701	0.62632	0.18556
4.5	0.19420	0.64533	0.17592
5.0	0.18355	0.66170	0.16773

TABLE ( 12.1 )

CARSON'S SOLUTION

TIME (P.U.)	X = 0 (P.U.) CURRENT (P.U.)	X = 1 (P.U.)	
		VOLTAGE (P.U.)	CURRENT (P.U.)
0.0	1.00000	0.0	0.0
0.5	0.64435	0.0	0.0
1.0	0.46498	0.36788	0.36788
1.5	0.36673	0.44550	0.29849
2.0	0.30791	0.50154	0.25751
2.5	0.26955	0.54392	0.23074
3.0	0.24259	0.57721	0.21168
3.5	0.22246	0.60419	0.19715
4.0	0.20672	0.62659	0.18552
4.5	0.19396	0.64557	0.17588
5.0	0.18334	0.66192	0.16769

TABLE ( 12.2 )

EXAMPLE SOLUTION ( 1 )



TIME (P.U.)	X = 0 (P.U.) CURRENT (P.U.)	X = 1 (P.U.)	
		VOLTAGE (P.U.)	CURRENT (P.U.)
0.0	1.00000	0.0	0.0
0.5	0.60449	0.0	0.0
1.0	0.42182	0.36788	0.36788
1.5	0.32979	0.46357	0.29945
2.0	0.27828	0.52395	0.25828
2.5	0.24608	0.56605	0.23118
3.0	0.22388	0.59774	0.21188
3.5	0.20734	0.62290	0.19724
4.0	0.19431	0.64366	0.18558
4.5	0.18362	0.66123	0.17597
5.0	0.17461	0.67640	0.16783

TABLE ( 12.3 )

EXAMPLE SOLUTION ( 2 )

TIME (P.U.)	X = 0 (P.U.) CURRENT (P.U.)	X = 1 (P.U.)	
		VOLTAGE (P.U.)	CURRENT (P.U.)
0.0	1.00000	0.0	0.0
0.5	0.59928	0.0	0.0
1.0	0.41580	0.36788	0.36788
1.5	0.32418	0.46309	0.29815
2.0	0.27329	0.52285	0.25640
2.5	0.24166	0.56439	0.22906
3.0	0.21992	0.59562	0.20966
3.5	0.20374	0.62042	0.19499
4.0	0.19099	0.64088	0.18333
4.5	0.18053	0.65821	0.17374
5.0	0.17170	0.67317	0.16563

TABLE ( 12.4 )

EXAMPLE SOLUTION ( 3 )

## APPENDIX (12.1)

IMPULSE RESPONSE OF A SEMI-INFINITE  
SINGLE CIRCUIT TRANSMISSION LINE

The T.L.E.s of a single circuit transmission line are:

$$-\frac{\partial v(x,t)}{\partial x} = L \frac{\partial i(x,t)}{\partial t} + R \cdot i(x,t)$$

$$-\frac{\partial i(x,t)}{\partial x} = C \frac{\partial v(x,t)}{\partial t} + G \cdot v(x,t)$$

Taking Laplace transforms with respect to time and assuming the line to be initially at rest gives:

$$\begin{aligned} -\frac{dV(x,s)}{dx} &= (sL + R)I(x,s) \\ -\frac{dI(x,s)}{dx} &= (sC + G)V(x,s) \end{aligned} \tag{12.15}$$

from which the equations in terms of one variable can be deduced:

$$\begin{aligned} \frac{d^2 V(x,s)}{dx^2} &= (sL + R)(sC + G)V(x,s) \\ \text{or } \frac{d^2 V(x,s)}{dx^2} &= \gamma^2 V(x,s) \\ \frac{d^2 I(x,s)}{dx^2} &= \gamma^2 I(x,s) \end{aligned} \tag{12.16}$$

(a) Voltage response for a unit applied impulse of voltage

Equation (12.16) has the solution:

$$V(x,s) = e^{-\gamma x} \tag{12.17}$$

$$\begin{aligned}\text{Now } \gamma &= \sqrt{(sL + R)(sC + G)} \\ &= \sqrt{LC} \sqrt{(s + \rho)^2 - \sigma^2}\end{aligned}$$

where

$$\rho = \frac{1}{2}\left(\frac{R}{L} + \frac{G}{C}\right), \quad \sigma = \frac{1}{2}\left(\frac{R}{L} - \frac{G}{C}\right)$$

Thus:

$$V(x, s) = e^{-x\sqrt{LC} \sqrt{(s + \rho)^2 - \sigma^2}} \quad (12.18)$$

The impulse response is given by the inverse Laplace transform of equation (12.18). This is derived in APPENDIX (12.2) and gives the response:

$$v(x, t) = e^{-\rho t} \left\{ \delta\left(t - \frac{x}{a}\right) + \frac{\sigma x}{a} \frac{I_1(\sigma p)}{p} \right\} u\left(t - \frac{x}{a}\right) \quad (12.19)$$

where  $a = \frac{1}{\sqrt{LC}}$ ,  $p = \sqrt{t^2 - \left(\frac{x}{a}\right)^2}$ ,  $\delta$  is a Dirac delta function  $u$  a unit step function and  $I_1$  a modified Bessel function of the first kind.

(b) Current response for a unit applied impulse of voltage

From equations (12.15) and (12.16) the current response is:

$$I(x, s) = (sC + G) \cdot \frac{e^{-x\sqrt{LC} \sqrt{(s+\rho)^2 - \sigma^2}}}{\sqrt{LC} \sqrt{(s+\rho)^2 - \sigma^2}}$$

The inverse Laplace transform of this function is obtained in APPENDIX (12.2) giving the current response:

$$i(x, t) = \frac{e^{-\rho t}}{Z_c} \left\{ \delta\left(t - \frac{x}{a}\right) - \sigma \left[ I_0(\sigma p) - t \frac{I_1(\sigma p)}{p} \right] \right\} u\left(t - \frac{x}{a}\right) \quad (12.20)$$

where  $z_c = \sqrt{\frac{L}{C}}$  and  $I_0$  is a modified Bessel function of the first kind.

(c) Current response for a unit applied impulse of current

Since both current and voltage propagation are governed by the same equation, (12.16), the required response will be given by equation (12.19) written for current instead of voltage.

(d) Voltage response for a unit applied impulse of current

As in part (b) the voltage response can be shown to be:

$$V(x,s) = (sL + R) \frac{e^{-x\sqrt{LC}} \sqrt{(s+\rho)^2 - \sigma^2}}{\sqrt{LC} \sqrt{(s+\rho)^2 - \sigma^2}}$$

the inverse of which is derived in APPENDIX (12.2) giving the response:

$$v(x,t) = z_c e^{-\rho t} \left\{ \delta\left(t - \frac{x}{a}\right) + \sigma \left[ I_0(\sigma p) + t \frac{I_1(\sigma p)}{p} \right] \right\} u\left(t - \frac{x}{a}\right) \quad (12.21)$$

## APPENDIX (12.2)

INVERSE LAPLACE TRANSFORMS

(a) Inverse transform of  $P(s) = e^{-K\sqrt{(s+\rho)^2 - \sigma^2}}$

$$\mathbf{L}^{-1}\{P(s)\} = \mathbf{L}^{-1}\{F(s+\rho)\} = e^{-\rho t} f(t)$$

where  $F(s) = e^{-K\sqrt{s^2 - \sigma^2}}$

Note also that  $\mathbf{L}^{-1}\left\{\frac{d}{ds} F(s)\right\} = -t \cdot f(t)$

$$\begin{aligned} \text{Now } \frac{d}{ds} F(s) &= -K \cdot s \cdot \frac{e^{-K\sqrt{s^2 - \sigma^2}}}{\sqrt{s^2 - \sigma^2}} \\ &= -K \cdot s \cdot G(s) \end{aligned}$$

From tables of Laplace transforms (ABRAMOWITZ, 1968):

$$\begin{aligned} \mathbf{L}^{-1}\{G(s)\} &= I_0(\sigma\sqrt{t^2 - K^2}) \cdot u(t-K) \\ &= g(t) \end{aligned}$$

Now  $\mathbf{L}^{-1}\{s \cdot G(s) - g(0_+)\} = \frac{d}{dt} g(t)$

$$g(0_+) = \lim_{s \rightarrow \infty} \{s \cdot G(s)\} = 0$$

so that  $\mathbf{L}^{-1}\{s \cdot G(s)\} = \frac{d}{dt} g(t)$

$$\begin{aligned} \frac{d}{dt} g(t) &= u(t-K) \frac{d}{dt} I_0(\sigma\sqrt{t^2 - K^2}) + I_0(\sigma\sqrt{t^2 - K^2}) \frac{d}{dt} u(t-K) \\ &= \frac{\sigma t}{\sqrt{t^2 - K^2}} I_1(\sigma\sqrt{t^2 - K^2}) + \delta(t-K) \quad \text{for } t \geq K \end{aligned}$$

where  $I_1$  and  $I_0$  are modified Bessel functions of the first kind.

$$-K \frac{d}{dt} g(t) = -t \cdot f(t)$$

$$\therefore f(t) = \left\{ \frac{K}{t} \delta(t-K) + K\sigma \frac{I_1(\sigma\sqrt{t^2-K^2})}{\sqrt{t^2-K^2}} \right\} u(t-K)$$

which gives the inverse transform

$$\mathbf{L}^{-1}\{P(s)\} = e^{-\rho t} \left\{ \delta(t-K) + \frac{K\sigma}{\sqrt{t^2-K^2}} I_1(\sigma\sqrt{t^2-K^2}) \right\} u(t-K) \quad (12.22)$$

$$(b) \quad \text{Inverse transform of } P(s) = (sC+G) \frac{e^{-K\sqrt{(s+\rho)^2-\sigma^2}}}{H\sqrt{(s+\rho)^2-\sigma^2}}$$

This transform is easily found from the transforms derived in part (a) of this appendix.

$$\begin{aligned} \mathbf{L}^{-1} \left\{ \frac{e^{-K\sqrt{(s+\rho)^2-\sigma^2}}}{\sqrt{(s+\rho)^2-\sigma^2}} \right\} &= e^{-\rho t} I_0(\sigma\sqrt{t^2-K^2}) \cdot u(t-K) \\ &= g(t) \end{aligned}$$

$$\text{Now } \lim_{s \rightarrow \infty} \{s \cdot G(s)\} = 0$$

$$\text{so that } \mathbf{L}^{-1}\{s \cdot G(s)\} = \frac{d}{dt} g(t)$$

$$\begin{aligned} \frac{d}{dt} g(t) &= e^{-\rho t} \frac{d}{dt} \{I_0(\sigma\sqrt{t^2-K^2}) u(t-K)\} \\ &\quad + I_0(\sigma\sqrt{t^2-K^2}) u(t-K) \frac{d}{dt} e^{-\rho t} \\ &= e^{-\rho t} \left\{ \delta(t-K) + \frac{\sigma t}{\sqrt{t^2-K^2}} I_1(\sigma\sqrt{t^2-K^2}) - \rho I_0(\sigma\sqrt{t^2-K^2}) \right\} \end{aligned}$$

for  $t \geq K$

and

$$\begin{aligned}\mathbf{L}^{-1}\{P(s)\} &= e^{-\rho t} \left\{ \frac{C}{H} \delta(t-K) + \left( \frac{G}{H} - \frac{C\rho}{H} \right) I_0(\sigma\sqrt{t^2-K^2}) \right. \\ &\quad \left. + \frac{C\sigma t}{H\sqrt{t^2-K^2}} I_1(\sigma\sqrt{t^2-K^2}) \right\}\end{aligned}$$

for  $t \geq K$

Substituting  $H = \sqrt{LC}$  gives:

$$\begin{aligned}\mathbf{L}^{-1}\{P(s)\} &= \frac{e^{-\rho t}}{Z_c} \left\{ \delta(t-K) - \sigma [I_0(\sigma\sqrt{t^2-K^2}) - t \frac{I_1(\sigma\sqrt{t^2-K^2})}{\sqrt{t^2-K^2}}] \right\} \\ &\text{for } t \geq K\end{aligned}\tag{12.23}$$

where  $Z_c = \sqrt{\frac{L}{C}}$ ,  $\rho = \frac{1}{2}(\frac{R}{L} + \frac{G}{C})$  and  $\sigma = \frac{1}{2}(\frac{R}{L} - \frac{G}{C})$

$$(c) \quad \text{Inverse transform of } P(s) = (sL+R) \frac{e^{-K\sqrt{(s+\rho)^2-\sigma^2}}}{H\sqrt{(s+\rho)^2-\sigma^2}}$$

From part (b) this can be written immediately as:

$$\begin{aligned}\mathbf{L}^{-1}\{P(s)\} &= e^{-\rho t} \left\{ \frac{L}{H} \delta(t-K) + \left( \frac{R}{H} - \frac{L\rho}{H} \right) I_0(\sigma\sqrt{t^2-K^2}) \right. \\ &\quad \left. + \frac{L\sigma t}{H\sqrt{t^2-K^2}} I_1(\sigma\sqrt{t^2-K^2}) \right\}\end{aligned}$$

for  $t \geq K$

Substituting  $H = \sqrt{LC}$  gives:

$$\begin{aligned}\mathbf{L}^{-1}\{P(s)\} &= Z_c e^{-\rho t} \left\{ \delta(t-K) + \sigma [I_0(\sigma\sqrt{t^2-K^2}) + t \frac{I_1(\sigma\sqrt{t^2-K^2})}{\sqrt{t^2-K^2}}] \right\} \\ &\text{for } t \geq K.\end{aligned}\tag{12.24}$$



## APPENDIX (12.3)

WEIGHTING FUNCTIONS AND THEIR DERIVATIVESAS USED IN SECTION 12.2.3)

The examples given in section 12.2.3) were all solved in terms of the series resistance per unit system. Apart from forming a convenient basis for comparison with other solutions the per unit system also has the advantage of simplifying the weighting functions and their derivatives.

Per unit system applied to convolution integrals

Consider the convolution integral:

$$f(x,t) = \int_{\frac{x}{a}}^t W(x,T) v_0(t-T) dT \quad (12.25)$$

For the series resistance per unit system  $x = X_1 \cdot x_{p.u.}$ ,

$t = T_1 \cdot t_{p.u.}$  and  $T_1 = X_1/a$ . Substituting into (12.25) gives:

$$f(x,t) = \int_{\frac{X_1}{a} x_{p.u.}}^{T_1 t_{p.u.}} W(X_1 x_{p.u.}, T_1 T_{p.u.}) \times \\ v_0(T_1 t_{p.u.} - T_1 T_{p.u.}) d(T_1 T_{p.u.})$$

$X_1$  and  $T_1$  are constants enabling  $f$  to be obtained in terms of p.u. from:

$$f(x_{p.u.}, t_{p.u.}) = \int_{x_{p.u.}}^{t_{p.u.}} T_1 u(x_{p.u.}, T) v_0(t_{p.u.} - T) dT \quad (12.26)$$

where  $u(x_{p.u.}, T) = W(x_1 x_{p.u.}, T_1 T)$

For the series resistance line  $T_1 u(x_{p.u.}, t_{p.u.})$  will be a simpler function than  $W(x, t)$ .

### Weighting Functions: $T_1 u(x, t)$

(a)  $t > x$

$$\begin{aligned}
 T_1 u_1(x, t) &= \frac{e^{-t}}{2} \left\{ (t+x) \frac{I_1(p)}{p} - I_0(p) \right\} \\
 T_1 u_2(x, t) &= \frac{e^{-t}}{2} \left\{ (t+x) \frac{I_1(p)}{p} + I_0(p) \right\} \\
 T_1 u_3(x, t) &= \frac{e^{-t}}{2} \left\{ I_0(p) - (t-x) \frac{I_1(p)}{p} \right\} \\
 T_1 u_4(x, t) &= \frac{e^{-t}}{2} \left\{ I_0(p) + (t-x) \frac{I_1(p)}{p} \right\}
 \end{aligned} \tag{12.27}$$

where  $p = +\sqrt{t^2 - x^2}$

(b)  $t = x$ . For  $t = x$  the weighting functions are obtained as their limiting forms when  $p \rightarrow 0$ .

$$\begin{aligned}
 \lim_{p \rightarrow 0} T_1 u_1(x, t) &= \frac{e^{-t}}{2} \left\{ \frac{t+x}{2} - 1 \right\} \\
 \lim_{p \rightarrow 0} T_1 u_2(x, t) &= \frac{e^{-t}}{2} \left\{ \frac{t+x}{2} + 1 \right\} \\
 \lim_{p \rightarrow 0} T_1 u_3(x, t) &= \frac{e^{-t}}{2} \\
 \lim_{p \rightarrow 0} T_1 u_4(x, t) &= \frac{e^{-t}}{2}
 \end{aligned} \tag{12.28}$$

Time Derivatives of Weighting Functions:  $\frac{d}{dt} (T_1 u(x, t))$

(a)  $t > x$

$$\frac{d}{dt}(T_1 u_1(x, t)) = \frac{e^{-t}}{2} \left[ (1-t) \frac{I_1(p)}{p} + \frac{t}{t-x} I_2(p) \right] - T_1 u_1(x, t)$$

$$\frac{d}{dt}(T_1 u_2(x, t)) = \frac{e^{-t}}{2} \left[ (1+t) \frac{I_1(p)}{p} + \frac{t}{t-x} I_2(p) \right] - T_1 u_2(x, t)$$

$$\frac{d}{dt}(T_1 u_3(x, t)) = \frac{e^{-t}}{2} \left[ (t-1) \frac{I_1(p)}{p} - \frac{t}{t+x} I_2(p) \right] - T_1 u_3(x, t)$$

$$\frac{d}{dt}(T_1 u_4(x, t)) = \frac{e^{-t}}{2} \left[ (t+1) \frac{I_1(p)}{p} + \frac{t}{t+x} I_2(p) \right] - T_1 u_4(x, t)$$

(12.29)

where  $I_2(p)$  is a modified Bessel function of the first kind and is related to  $I_1(p)$  and  $I_0(p)$  through:

$$I_2(p) = I_0(p) - \frac{2I_1(p)}{p}$$

(b)  $t = x$ . As before the derivatives are found as the limiting case when  $p \rightarrow 0$ .

$$\lim_{p \rightarrow 0} \frac{d}{dt} (T_1 u_1(x, t)) = \frac{e^{-t}}{2} \left[ \frac{(1-t)}{2} - \frac{t(t+x)}{8} \right] - \frac{e^{-t}}{2} \left[ \frac{t+x}{2} - 1 \right]$$

$$\lim_{p \rightarrow 0} \frac{d}{dt} (T_1 u_2(x, t)) = \frac{e^{-t}}{2} \left[ \frac{(1+t)}{2} + \frac{t(t+x)}{8} \right] - \frac{e^{-t}}{2} \left[ \frac{t+x}{2} + 1 \right]$$

$$\lim_{p \rightarrow 0} \frac{d}{dt} (T_1 u_3(x, t)) = \frac{e^{-t}}{2} \left[ \frac{t-1}{2} - 1 \right]$$

$$\lim_{p \rightarrow 0} \frac{d}{dt} (T_1 u_4(x, t)) = \frac{e^{-t}}{2} \left[ \frac{t+1}{2} - 1 \right]$$

(12.30)

## SECTION 13

CONCLUSIONS

The constant coefficients T.L.E.s form the simplest mathematical representation of a transmission line available to electrical engineers. In the power systems context such a representation ignores the transient effects of skin effect on line conductors, finitely conducting ground and corona discharge. In spite of these shortcomings, this representation is still frequently used in power system transient propagation problems when voltage levels are below those of corona breakdown, when maximum crest voltages are of primary concern, or when the complexity is such that a simple representation is required to reduce work to a practical engineering basis.

Of the various forms of line that can be modelled by the constant coefficient T.L.E.s the series resistance line is most representative of power system transmission lines although low loss or distortionless lines offer a representation suitable for short term transient problems. Of all the line types able to be represented by the constant coefficient T.L.E.s only lines with dispersionless transmission (lossless or distortionless components of propagation) yield exact and generally applicable solutions describing propagation of signals along them. Thus even for a linear representation of a power system transmission line, approximate methods of solution must often be resorted to if any other than short term transient effects are being considered. As a consequence many different solution procedures for solving transient propagation problems have

evolved. Of these, those that use line variables directly and solve in the time domain combine best, a versatility of application to transient problems and the ability to be extended to non-linear transmission line representation.

Prior to this analysis finite difference techniques and discrete lumped loss approximations were the only solution methods of the above type available for general application to transient propagation problems on lossy transmission lines. This analysis has considered in detail, application of the M.O.C. to the problem of propagation on lossy transmission lines and has produced solutions that are demonstrably superior to either of the two alternatives listed above. Of the M.O.C. this study has shown that:

1. It can be applied to more than just lossless lines.
2. It can be applied to multiple circuit transmission lines directly.
3. Its direct application to multiple circuit transmission lines is independent of transmission line geometry or any combination of line parameters.
4. Exact solutions are obtained for transmission lines whose components of propagation are dispersionless.
5. Where exact solutions are not possible, approximate solutions of high accuracy can be obtained.
6. The solution equations are simple and of a form easily handled by a digital computer.

7. The characteristic equations of the T.L.E.s can be constructed using standard matrix procedures.
8. It can be used in conjunction with modal components to solve indirectly for multiple circuit transmission lines. When formulated in a form allowing direct solution between any two points on a transmission line it proves particularly effective in this application.
9. For distortionless lines the exact solution produced can be implemented graphically. This gives a new graphical solution capable of including transmission line attenuation thus enabling graphical solutions of short term transient problems to be treated with reasonable accuracy.

When compared with the finite difference and discrete lumped loss approximations the M.O.C.:

10. Does not suffer from the problem of numerical instability as do finite difference methods.
11. Is shown to include the discrete lumped loss approximation as a special case of a wider class of approximate solutions derived from its application. For the series resistance line the discrete lumped loss approximation is shown to be the least accurate of the alternative solutions considered.

The Method of Characteristics has thus shown itself to be a viable way of solving transmission problems and to have a range of application much wider than has been generally recognised. In the power systems context it has shown a

particular suitability to the solution of transient problems where it can cope easily with arbitrary switching signals and non-linear terminal equipment. It is particularly suited for use on digital computers where it can be used for any signals on lossless or lossy transmission lines consisting of single or multiple mutually coupled circuits.

## APPENDIX (13.1)

PER UNIT SYSTEM FOR A SERIES RESISTANCE LINE (HEDMAN, 1971)

This per unit system given by HEDMAN enables a more general comparison of solutions. In this system:

$$X_1 = 1 \text{ p.u. distance} = \frac{2Z_c}{R}$$

$$T_1 = 1 \text{ p.u. time} = \frac{X_1}{a} = \frac{2L}{R}$$

$$Z_{c1} = 1 \text{ p.u. surge impedance} = 1 \text{ ohm}$$

$$R_1 = 1 \text{ p.u. series resistance} = 2 \text{ ohms}$$

In this system, an applied unit step of voltage will travel 1 p.u. distance in 1 p.u. time and will have a magnitude of  $e^{-1}$  at the toe of the wave.

For a transmission line corresponding to the line component of the symmetrical two circuit transmission line used in section 11)  $T_1$  and  $X_1$  have the values  $4.541 \times 10^{-2}$  seconds and  $1.314 \times 10^7$  metres respectively.



## APPENDIX (13.2)

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P A R T    2

TRANSMISSION LINE ANALYSIS WITH FREQUENCY  
DEPENDENT PARAMETERS

## SECTION 14

INTRODUCTION TO TRANSMISSION LINE ANALYSIS  
WITH FREQUENCY DEPENDENT PARAMETERS

In 1926 CARSON commented in his now classic paper: "The problem of wave propagation along a transmission system composed of an overhead wire parallel to the surface of the earth, in spite of its great technical importance, does not appear to have been satisfactorily solved". In 1972 TRAN commented in his paper: "The problem of analyzing the earth effects on transmission lines has been studied for a long time but it is not yet satisfactorily resolved from an engineering point of view". This problem has received a great deal of attention over an extended period of time and continues to do so.

The effect of the finite conductivity of the earth on the propagation of signals along transmission lines has been neglected in previous sections of this thesis. In real situations the earth is finitely conducting. When it becomes part of the transmission circuit the properties of propagation are greatly altered. Whereas the line parameters are constant for infinite conductivity they become functions of frequency for finite conductivity. Overhead conductors also have finite conductivity but their effect on signal propagation is much smaller on account of their being better conductors and having small cross-sections.

#### 14.1) OBJECTS AND SCOPE OF ANALYSIS

Literature on this topic is both considerable and dispersed. It is intended to alleviate this problem by collecting together most of the significant work that has been carried out to date. The references given do not make up a complete bibliography of the subject and the analysis should be regarded as a summary of significant developments and results. In presenting this work enough detail has been included to allow a feel for the subject to be obtained in analytical method, results, their form and physical properties. By annotation and extension where desired this analysis should readily extend to a comprehensive treatise giving a detailed and advanced background knowledge of any part of the subject area.

In spite of its being of a review nature this analysis does contain some original work, given in sections 15.4.2), 15.4.3) and 16.4.5).

#### 14.2) ABSTRACT OF ANALYSIS

The analysis consists of two sections, 15) and 16), together with an introduction, section 14), and a list of references given as an appendix to the analysis at the end of section 16).

Section 15) deals with formulating the effects of finite conductivity in transmission systems. On account of its greater effect, finite conductivity of the earth is given priority. Finite conductivity of the wires is not dealt with in detail. The major formulation methods are classified and

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discussed. Assumptions usually made are given together with any limitations that they introduce. A detailed résumé of formulations is given. The T.L.E.s are extended to include the effect of the earth's finite conductivity according to CARSON'S (1926) formulation.

Section 16) deals with solving travelling wave problems on transmission systems with frequency dependent parameters. The methods of solution are classified. Examples are presented and discussed.

#### 14.3) CONVENTIONS, ABBREVIATIONS AND DEFINITIONS

Where possible, the conventions used in each reference have been preserved when referring to it. Thus it should not be expected that a convention used in discussing one reference will automatically follow in the discussion of the next reference. The conventions used should be self evident. Otherwise they have been redefined as required.

Except where otherwise stated, the following symbols have been used according to the conventions given below:

$\epsilon = \epsilon_0 \epsilon_r$	permittivity
$\mu = \mu_0 \mu_r$	permeability
$\sigma$	conductivity
$\rho$	resistivity
$c$	speed of light
$h$	height of conductor above ground
$a$	conductor radius
$r_{12}$	the radial distance (i.e. the shortest distance) between conductors 1 and 2.



$r'_{12}$	the radial distance between conductor 1 and the image of conductor 2 with respect to the surface of the ground.
$\omega$	angular frequency (radians)
$f$	frequency (Hz)
$\gamma$	propagation constant
$\gamma_0$	propagation constant of free space
$C$	line capacitance
$L$	line inductance
$R$	line resistance
$G$	line conductance
$Z$	impedance - usually transmission line series impedance (e.g. $Z(\omega) = R + j\omega L$ )
$Y$	admittance - usually transmission line shunt admittance (e.g. $Y(\omega) = G + j\omega C$ )
$j$	$= \sqrt{-1}$
$J$	current density
$E$	electric field intensity

#### Mathematical functions

$\log$	natural logarithm (i.e. $\log_e$ )
$e^{(x)}$	exponential
$\text{erfc}(x)$	complementary error function
$\left. \begin{matrix} J_0 \\ J_1 \end{matrix} \right\}$	Bessel functions of the first kind, orders 0 and 1
$Y_1$	Bessel function of the second kind, order 1
$\left. \begin{matrix} I_0 \\ I_1 \end{matrix} \right\}$	Modified Bessel functions of the first kind, orders 0 and 1.

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$K_0$	} Modified Bessel functions of the second kind, orders
$K_1$	
	0 and 1.
$H_1$	Struve function, order 1
$\delta(t)$	Derac delta function
$H(t-T)$	} Unit step function
$U(t-T)$	

### Subscripts

c	denotes conductor
g	{ denotes ground
e	
	{ denotes earth

### Operators

p	} Operator symbols as used in operational analysis
s	
$\nabla^2$	$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})$

### Abbreviations

T.L.E.s	Transmission Line Equations
w.r.t.	with respect to.

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## SECTION 15

FORMULATION OF EFFECTS OF FREQUENCY DEPENDENT PARAMETERS

The treatment given surge propagation in the Linear Transmission Line Analysis was based on a perfectly conducting earth. In reality, however, the earth constitutes a relatively poor semi-infinite conductor whose close proximity to, or connection into, power transmission systems can have a very great effect on their transmission characteristics. For overhead transmission lines, the treatment based on a perfectly conducting earth is usually satisfactory as regards the calculation of surge impedances and crest voltages but the effect of the finite conductivity of the earth on attenuation and wavefront distortion cannot be neglected.

The effect on propagation of non-perfect conductors is frequently referred to as the 'Skin Effect'. When formulated in the frequency domain it gives rise to line parameters that are functions of frequency. For surge analysis in the time domain it corresponds to a sidewise diffusion of current density from the conductor surface into its surrounding cross section thus giving rise to parameters that are functions of time (MILLER, 1947).

Formulating the transmission properties associated with a finitely conducting earth has been the topic of much investigation, beginning in the early 1900's and continuing to the present time. This has resulted in a large volume of dispersed literature. The references presented here are mostly those that have had a long association with power systems analysis and in particular those associated with surge propagation in power transmission systems.

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One of the difficulties in formulating these effects is the extensive nature of the earth which in most cases requires the use of electromagnetic field theory rather than conventional transmission line circuit theory. Almost always the effects are formulated for steady state sinusoidal signals although occasionally operational expressions or time variable expressions are derived (Fig. 15.3). Most of the derivations listed here are concerned with transmission lines consisting of aerial conductors with an earth return although expressions have been derived for conductors that are buried in, or are on the surface of, the ground (SUNDE, 1968). For typical overhead transmission lines the shunt admittance is small and the propagation constant and characteristic impedance are affected by the finite conductivity of the earth mainly through its influence on the series impedance, (whereas the propagation constant of a buried conductor is affected by the finite conductivity of the earth, mainly through its influence on the admittance of the conductor). Consequently most of the references listed here are concerned with formulating the effect of the earth on the series impedance of the line (Fig. 15.3). When formulated, the effect of the earth on the shunt admittance of the line is shown, in the frequency range associated with power system transients, to be negligible (WISE, 1934) (ARISMUNANDAR, 1963).

15.1) METHODS OF FORMULATION

On examining the many formulations of earth effects that have appeared, four approaches emerge as dominant. For the sake of classification these have been titled as shown below together with the authors who have used them.

(1) Faraday's equation method	{	CARSON 1926
		WISE 1931
		BROWN 1971
		ROBERT 1969
(2) Current element method	{	WISE 1934
		WISE 1948
	{	ARISMUNANDAR 1963
		KAIDANOV 1965
(3) Multiple transform method	{	MULLINEUX 1965
		WEDEPOHL 1966
		THESIS - Section 15.4)
(4) Diffusion equation method	{	MILLER 1947
		RUDENBERG 1969
		* SHAH 1971

\* SHAH'S formulation has been so classified because it ignores the effect of the earth current distribution on the magnetic field in the air (Section 15.2.5)).

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### 15.1.1) Faraday's Equation Method

This method was first proposed by CARSON in 1926. WISE, in 1931, repeated CARSON'S analysis but took the ground permeability into account. This was done by going through CARSON'S paper and writing in the permeability wherever CARSON had replaced it by unity. In BROWN 1971, we have the clearest statement of the method (given in an appendix to his article) where it is outlined in some detail for the case of a single overhead conductor with an earth return. BROWN also includes the effect of the ground permeability (see Fig. (15.2)). ROBERT 1969, in deriving his operational expressions has used this method insofar as he used an operational equivalent of CARSON'S result as a starting point.

In this approach the axial electric field intensity along the surface of the conductor is found by applying the equation  $\nabla \times E = - \frac{\partial B}{\partial t}$  to a plane surface of differential axial length positioned between the conductor and the ground immediately beneath it. This is shown in Fig. (15.1).

The line integral  $\oint E \cdot dl$  is found around the perimeter of this surface at the same time as the surface integral  $\iint H \cdot \hat{n} dA$  is found across the surface. This necessitates expressions for the electric field intensity along the surface of the ground and the magnetic field intensity in the air. The electric field in the ground and the magnetic field due to ground currents in the air are found by writing an integral solution of the wave equation in both air and ground. The resulting arbitrary functions are eliminated by matching the magnetic field intensity and flux density across the surface of the earth.

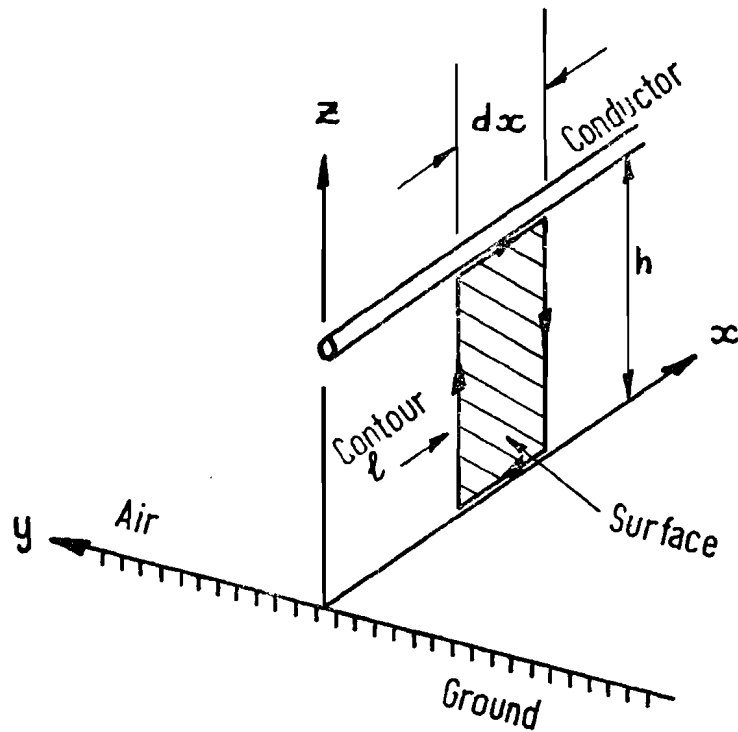


Fig. (15.1) Surface to which  $\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}$  is applied.

The total magnetic field in the air is then found as the sum of that due to the distribution of ground currents and that due to the conductor currents. Assuming exponential propagation along the transmission line and constant transmission line shunt admittance then enables the propagation constants (thence the series impedances) to be found by equating the electric field intensities at the surfaces of the conductors.

### 15.1.2) Current Element Method

This is the formulation method which has been used most. It is comprehensively presented by SUNDE (1968) in his book where the whole problem of earth conduction effects in transmission systems is dealt with in considerable detail.

In this method the field in air is described by a vector wave function (usually the Hertz vector) of a current element dipole of differential length above a finitely conducting ground. This wave function is derived in a manner similar to that already described in section 15.1.1) with the wave function being written in integral form in ground and air and the resulting arbitrary functions being derived by matching the fields across the system boundaries. The wave function for the current distribution along the conductor is then obtained by integrating the wave functions for the current element dipoles over the length of the conductor.

WISE in his papers (1934 and 1948) uses a wave function as derived by H. VON HOERSCHELMANN (1912) which ignores the effect of ground permeability. KAIDANOV (1965) uses the field solution as derived by SOMMERFELD. The field due to a horizontal current element dipole above a uniform earth as derived by Sommerfeld can be seen in SOMMERFELD (1964). SUNDE (1968) derives the vector wave function for a horizontal current element dipole above an earth with horizontal stratification and exponential variation of earth properties while ARISMUNANDAR (1963) uses SUNDE'S method to derive the field for a uniform earth.



The electric field intensity in air is readily deduced from the vector wave function and the self and mutual impedances of the transmission circuits are found by further integrating this axially along the lengths of the conductors. For infinite lines the integration is carried out from plus to minus infinity as done by WISE (1934 and 1948). SUNDE (1968) considers both finite and infinite cases.

#### 15.1.3) Multiple Transform Method

This is the formulation method that has appeared most recently. It was first proposed by MULLINEUX in 1965 where it was used to produce an expression equivalent to that of CARSON'S but with the effect of ground permeability included. In 1966 it was used by WEDEPOHL to produce formulas for the effects of horizontally stratified earth structures where both displacement currents in the ground and ground permeability were accounted for (Fig. (15.2)). It is used in section 15.4) to derive expressions for earth effects in both frequency and time under the assumptions made by MULLINEUX (1965).

Here the wave equation in terms of the electric field intensity is written in both air and ground. Taking integral transforms with respect to time and space coordinates converts the set of partial differential equations to a set of algebraic equations the coefficients of which are determined by use of the transformed boundary conditions on the system interfaces. The algebraic equations are solved to find a transformed expression for the electric field in air. Inverse transformations are then carried out replacing the

independent variables until the untransformed solution is obtained. Having thus obtained the electric field intensity in air, the series impedance is obtained in the manner outlined in sections 15.1.1) and 15.1.2).

#### 15.1.4) Diffusion Equation Method

For a conducting medium, neglecting displacement currents in the wave equation leads to the diffusion equation. This is often done to simplify the mathematical manipulation required in solving field problems. It is a good approximation only for good conductors or for low frequency signals (section 15.2.1)). Solving the resulting diffusion equation leads to the distribution of current density in the conductor from which the electric field intensity in the conductor and the internal impedance of the conductor can be obtained. The method is in itself only capable of supplying expressions for the internal impedances of the conductors. Its application to transmission circuits as the sole correction for finite conductivity is therefore valid only when the conductor cross sections are much smaller than their separations (a point completely lost on SHAH (1971)), or where there are good conductors and coaxial symmetry.

MILLER (1947) solves for the distribution of current density across the conductor in response to a suddenly applied step of current. His solutions are in the time domain and produce transient conductor resistances and internal inductances that are functions of time. Insofar as he considers only good conductors and their internal impedances his application is valid and his work provides an interesting insight to the physical effects of finite

conductivity. RUDENBERG (1969), like SHAH, applies the method to a conductor with an earth return but in assuming coaxial symmetry eases the effects of its restriction.

#### 15.2) ASSUMPTIONS AND THEIR LIMITATIONS

For most practical problems the accuracy of a rigorous solution for signal propagation in the presence of a finitely conducting earth is not required. Furthermore, in view of the probable uncertainty with which the earth parameters - resistivity, permittivity, permeability - will be known, it will not be justified. It has therefore been usual to include simplifying assumptions in formulating the effects of finite conductivity. Not all problems, however, can tolerate the same degree of approximation and the application of certain formulations is limited as a result.

A summary of the major assumptions made in the various formulations is given diagrammatically in Fig. (15.2). In all except two cases these assumptions are limited to statements about the conducting medium (earth). In the two exceptional cases, as well as making statements about the conducting medium, statements are also made about the magnetic field in air.

In his classic formulation CARSON (1926) assumed the ground to be a homogeneous, semi-infinite conductor. In this ground he further assumed that the relative permeability was unity, that displacement currents could be neglected and that the electric field intensity existed only in a direction parallel to the overhead conductors. Of these assumptions

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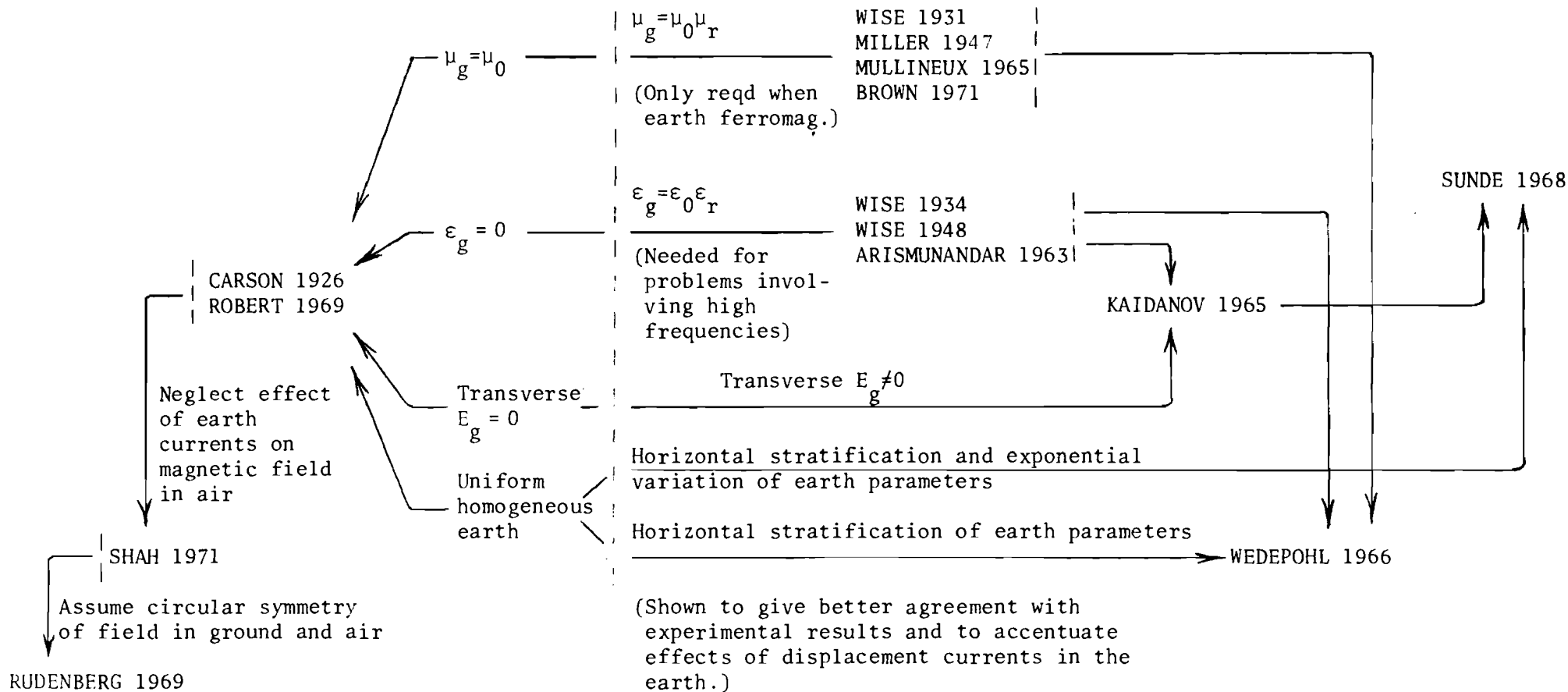


Fig. (15.2) Major assumptions of references listed in formulation. (Subscript 'g' refers to ground.) The further to the right on the diagram, the more assumptions have been removed.

only two are severe in limiting the application of his formulation. These are that the earth is homogeneous and that displacement currents can be neglected.

#### 15.2.1) Neglecting Displacement Currents in the Ground

The wave equation for a conducting medium in the time domain is:

$$\nabla^2 E = \mu\sigma \frac{\partial E}{\partial t} + \mu\epsilon \frac{\partial^2 E}{\partial t^2} \quad (15.1)$$

which for a steady state sinusoidal travelling wave becomes:

$$\nabla^2 E = j\omega\mu(\sigma + j\omega\epsilon)E \quad (15.2)$$

Neglecting displacement currents reduces these equations to

$$\nabla^2 E = \mu\sigma \frac{\partial E}{\partial t} \quad (15.3)$$

and

$$\nabla^2 E = j\omega\mu\sigma E \quad (15.4)$$

Equation (15.3) is the diffusion equation and is a valid approximation to equation (15.1) only when  $\epsilon \ll \sigma$  and  $\left| \frac{\partial E}{\partial t} \right| \ll \left| \frac{\partial^2 E}{\partial t^2} \right|$ , i.e. for good conductors. This equation was solved by MILLER (1947) to obtain the conductor current density distribution as a function of time.

Equation (15.4) is a valid approximation to equation (15.2) when:

$$\frac{\omega\epsilon}{\sigma} = \rho\omega\epsilon \ll 1 \quad (15.5)$$

This limits the use of equation (15.4) to low frequencies. For a typical ground with say  $\rho = 1000\Omega\text{M}$  and  $\epsilon_r = 40$ , a

frequency of 4.5 KHz gives  $\rho\omega\epsilon \approx 0.01$ . For  $\rho = 100\Omega\text{M}$  a frequency of 45 KHz gives the same result. This is the major analytical limitation of CARSON'S formulation. It was removed by WISE (1934) to enable higher frequencies to be handled. Later formulations by ARISMUNANDAR (1963), SUNDE (1968) and KAIDANOV (1965) also removed this restriction. However, for steady state problems at typical power systems frequencies or even for slow power systems transients, this limitation is not likely to result in large errors.

Curves illustrating the effect on the series impedance correction terms of neglecting displacement currents are given by WEDEPOHL (1966) for both uniform and two-layer horizontally stratified earths. These clearly indicate that good results will be obtained only for low frequencies with the errors becoming accentuated in the two layer case.

#### 15.2.2) Assuming a Uniform Homogeneous Earth

In many respects this is the most severe of CARSON'S assumptions. SUNDE (1968) lists references compiling extensive experimental data on the mutual impedance of earth return circuits. This indicates that the earth cannot usually be regarded as a uniformly conducting medium but that with formulas based on a two-layer horizontal stratification of the earth, a satisfactory agreement with experimental results can be obtained much more frequently. Satisfactory results for the magnitude of series impedances can be obtained by using in formulas derived for a uniform earth, an equivalent resistivity based on a two layer earth. SUNDE (1968) derives curves for such equivalent resistivities where:

$$\rho_{\text{equivalent}} = \rho(\omega, \rho_1, \rho_2, d)$$

$\rho_1, \rho_2$  are the resistivities of the two layers and  $d$  is the thickness of the top layer.

This approach would, for example, give results suitable for problems involving mutual coupling between power lines and communication circuits where magnitude is of greater importance than phase angle.

WEDEPOHL (1966) gives curves for the series impedance correction factors calculated for two layer and uniform earths. These show clearly that marked differences can occur between the two, especially at low frequencies. For self-impedances the two layer results converge to the results for a uniform earth as the parameter  $r = 2h\sqrt{\omega\xi\sigma}$  increases. For mutual impedances this convergence is not nearly so apparent, especially where the upper layer resistivity is low. SUNDE (1968) shows in his two layer formulations that when the surface resistivity is high, its effect is fairly small unless it extends to great depths. His curves also show that for large separations between conductors the mutual inductance depends mainly on the lower layer resistivity.

#### 15.2.3) Assuming Unity Relative Permeability in the Ground

This assumption is usually of no consequence. It is, however, easy to include when required and was first included in CARSON'S formulation by WISE (1931). Here WISE simply repeated CARSON'S analysis writing in the permeability wherever CARSON had replaced it by unity. The major difficulty associated with its inclusion was its complicating effect on the asymptotic and series expansions used by WISE

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and CARSON to evaluate the infinite integral form of their correction terms. Today the integral would be evaluated numerically on a digital computer where it makes little difference whether ground permeability is included or not.

#### 15.2.4) Transverse Components of $E_g$ are Zero

This is the same as saying that steady state conditions obtain; that at any instant the field is described quite accurately by a field derived from stationary charges and constant currents equal to the instantaneous values of the variable charges and currents under consideration.

KAIDANOV (1965) shows that the errors introduced are small at typical power system frequencies and at frequencies normally encountered during power system transients.

#### 15.2.5) Neglecting the Effect of Ground Current Distribution on the Magnetic Field in Air

As already stated in section 15.1.4), this has been done by only two authors. It amounts to approximating the effects of finite conductivity entirely by the internal impedances of the conductors. It is an unnecessary assumption which in the case of a semi-infinite earth return achieves little in the way of simplification.

### 15.3) RÉSUMÉ OF FORMULATIONS

A summary of the formulations considered is given diagrammatically in Fig. (15.3). It is convenient when considering these formulations to categorize them into those derived for steady state sinusoidal signals and others. Most of the formulations have been carried out in terms of steady

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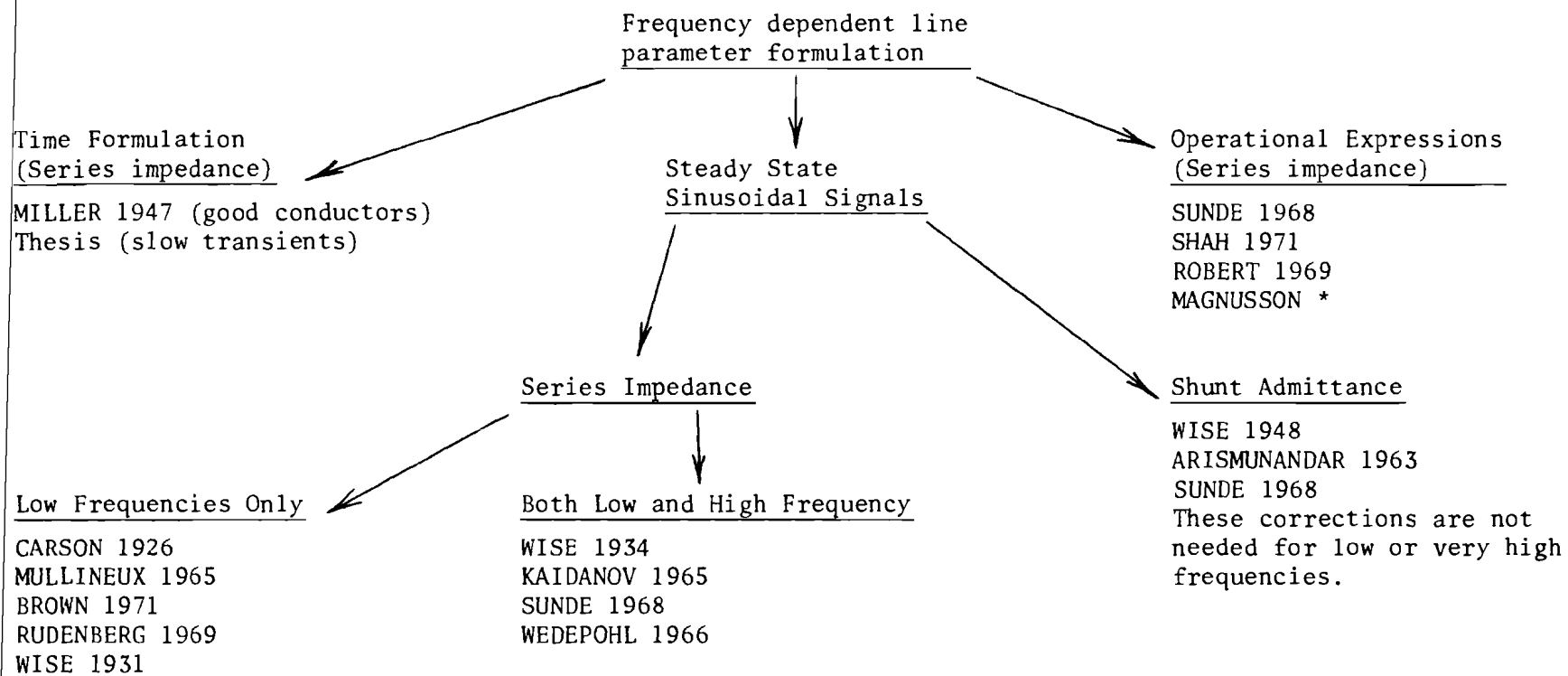


Fig. (15.3) Summary of formulations classified according to topic.

\* MAGNUSSON has not formulated ground effects but has derived operational expressions from the results of CARSON and others.

state sinusoidal signals and concern corrections to the transmission line's series impedance. In this form the results are used predominantly for constant frequency fault and load calculations, mutual coupling interference between power systems and communications circuits, propagation of carrier signals etc. The operational form is frequently used where transient problems are being considered, mostly to find the response of the transmission line to an applied step or impulse. Here the formulation is usually concerned with developing an earth corrected expression for the propagation constant. These formulations will now be considered in more detail. The axis convention used is shown in Fig. (15.1).

### 15.3.1) Steady State Sinusoidal Signals (Series Impedance)

#### (a) CARSON (1926)

Used Faraday's Equation Method to derive an integral correction term for the series impedance of a transmission line with an earth return. This term was divided into real and imaginary components to enable corrections to the series resistance and inductance to be found. These correction terms were expanded into asymptotic and series expansions for large and small values of their arguments to enable calculation. For the mid-range values of their arguments CARSON gave a series of curves which could be interpolated.

#### Results (Electromagnetic c.g.s. units)

$$\text{Self impedance } Z_s = R + j\omega 2\log \frac{2h}{a} + 4\omega J(2h', 0)$$

$$\text{Mutual impedance } Z_{12} = j\omega 2\log \frac{r'_{12}}{r_{12}} + 4\omega J(h'_1 + h'_2, y'_{12})$$

where

---

$$J(p,q) = P + jQ = \int_0^{\infty} (\sqrt{u^2 + j} - u) e^{-pu} \cos(qu) du$$

$$h = h'/\sqrt{\alpha} = \text{height of conductor above ground}$$

$$y = y'/\sqrt{\alpha} = \text{horizontal separation of conductors}$$

$$\alpha = 4\pi\sigma\omega$$

Asymptotic and series expressions for  $P$  and  $Q$  are derived in terms of the parameter  $r = \sqrt{p^2 + q^2}$ , the asymptotic expansions being valid for the range  $r > 5$  and the series being expanded for  $r \leq \frac{1}{4}$ .

#### Comment

It will be observed that the first two terms of the self impedance and the first term of the mutual impedance represent the series impedance if the ground is a perfect conductor; the infinite integral formulates the effect of the finite conductivity of the ground. It tends to zero as the conductivity tends to infinity. This expression is typical of all formulations that take the finite conductivity of conductors into account.

#### (b) WISE (1931)

Uses Faraday's Equation Method to derive an integral correction term for the series impedance of a transmission line with an earth return where the relative permeability of the ground can be greater than unity. His method of analysis and presentation of results are identical to that of CARSON (1926).

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### Results (Electromagnetic c.g.s. units)

$$\text{Self impedance } Z = z + j\omega 2\log \frac{2h}{a} + 4\omega J(2h', 0)$$

$$\text{mutual impedance } Z_{12} = j\omega 2\log \frac{r'_{12}}{r_{12}} + 4\omega J(h'_1 + h'_2, y'_{12})$$

where

$$J(p, q) = P + jQ = j\mu \int_0^{\infty} \frac{e^{-pT}}{\sqrt{T^2 + j} + \mu T} \cos(qT) dT$$

$$\text{and } \alpha = 4\pi\sigma\mu\omega$$

$z$  is the intrinsic impedance of the conductor.

Asymptotic and series expressions are derived for both  $P$  and  $Q$ . As the evaluation of the series formulas is rather laborious, tables for a range of values are given.

### (c) BROWN (1971)

Uses Faraday's Equation Method to derive the correction for a single conductor with an earth return, again taking the ground permeability into account.

### Result (M.K.S.A. units)

$$Z_s = R + \frac{j\omega\mu}{4\pi} \log \frac{2h-a}{a} + \frac{j\omega\mu}{2\pi} \int_0^{\infty} \frac{e^{-T(2h-a)}}{\sqrt{\eta^2 + T^2} + T} dT$$

$$\text{where } \eta^2 = j\omega\mu\sigma$$

The correction term is given as  $\Delta Z = \frac{\omega\mu}{\pi} [P + jQ]$  with

$$P = \int_0^{\infty} \frac{e^{-rT} \sin(rT)}{\sqrt{T^2 + 1} + 1} dT$$

$$Q = \int_0^{\infty} \frac{e^{-rT} \cos(rT)}{\sqrt{T^2 + 1} + T} dT$$

where  $r = (2h-a)\sqrt{\omega\mu\sigma}$

No series expressions are given for P and Q which when required are evaluated numerically.

(d) WISE (1934)

Here WISE used the Current Element Method to solve for the problem of a transmission line with an earth return where the earth displacement currents are not neglected. In this analysis WISE assumed the current to have an exponential distribution along the line ( $Ie^{-\gamma x}$ ); an assumption which in conjunction with an infinite line forces  $\gamma$  to be purely imaginary (lossless conductors). This enabled the current element dipoles to be integrated over the length of the infinite line. Using the propagation constant for free space can be regarded as a first approximation which leads to a corrected second approximation. However, as the second approximation for the propagation constant will be complex, a more refined approximation is unobtainable using this method. Hence this analysis presupposes reasonably efficient transmission.

Result (Electromagnetic c.g.s. units)

The effect of displacement currents in the ground, originally neglected in CARSON'S (1926) formulation can be included by substituting  $r\sqrt{1+j(\epsilon-1)/2c\lambda\sigma}$  in place of  $r$  in CARSON'S formulas.

The integral correction term is

$$J(p,q) = P+jQ = \frac{1}{s^2} \int_0^\infty (\sqrt{u^2+jS^2} - u) e^{-pu} \cos(qu) du$$


---

where  $S = \sqrt{1+j(\epsilon-1)/2c\lambda\sigma}$

and  $\lambda$  = wavelength of the signal (cm).

(e) RUDENBERG (1969)

Uses the Diffusion Equation Method to derive the series impedance of a single overhead conductor with an earth return. Here it is assumed that this can be represented by the coaxially symmetric arrangement shown in Fig. (15.4).

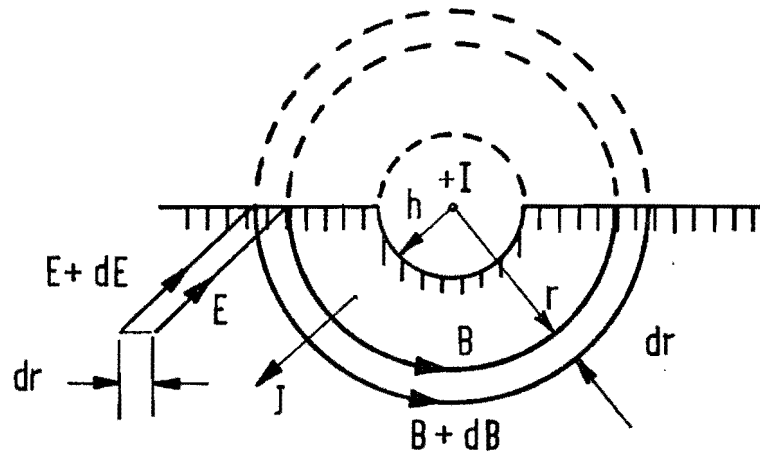


Fig. (15.4) RUDENBERG'S overhead line at a height 'h' above a finitely conducting ground return.

Applying Ampere's Law, Faraday's law and Ohms law produces the circularly symmetric diffusion equation

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial J}{\partial r} \right) = \frac{2\pi}{\rho} \frac{\partial J}{\partial t}$$

which is then solved in terms of Hankel functions giving the current density distribution in the ground enabling the internal impedance of the ground to be found. The total series impedance of the circuit is then determined by adding to the earth's internal impedance, the impedance of the circuit when the earth is assumed coaxial to the overhead

conductor and perfectly conducting.

### Results

(1) Internal impedance of the earth

(a) Low frequency  $R = \pi^2 f$ ,  $L = 2 \log \left( \frac{0.178}{h} \sqrt{\frac{\rho}{f}} \right)$ ,

$$E_{sg} = -I\omega \left[ \frac{\pi}{2} - j2 \log \left( \frac{0.178}{h} \sqrt{\frac{\rho}{f}} \right) \right]$$

(b) High frequency  $R = \frac{1}{h} \sqrt{2f\rho}$ ,  $L = \frac{1}{\sqrt{2}\pi h} \sqrt{\frac{\rho}{f}}$

$$E_{sg} = -I \cdot (1-j) \frac{\sqrt{2}}{h} \sqrt{f\rho}$$

$E_{sg}$  = ground surface electric field intensity directly beneath the conductor.

(2) Total circuit impedance

$$\text{Low frequency } L = 2 \left[ \log \left( \frac{4}{\gamma K d} \right) + \frac{1}{2} \right]$$

where  $K = 2\pi \sqrt{\frac{f}{\rho}}$ ,  $d$  = diameter of the conductor and

$\gamma$  = Eulers constant = 1.7811.

Multiplication of the above formulas by  $10^{-9}$  gives units of mH/Km and  $\Omega$ /Km.

### Comment

In the geometry chosen the field lines are only approximately concentric circles about the current carrying conductor but RUDENBERG feels that the effects of assuming circular symmetry are not greater in this case than those of other assumptions.

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(f) MULLINEUX (1965)

This reference introduces the most recent method of formulating the effects of finite conductivity. In it MULLINEUX uses the Multiple transform method to derive CARSON'S correction with the permeability of the ground included. This paper is expanded fully in section 15.4) where MULLINEUX'S formulation is further transformed into the time domain.

Result (M.K.S.A. units)

The correction term for the electric field intensity in air of a group of 'S' conductors with an earth return is:

$$\Delta E_x(y,z) = \frac{j\omega\mu_0}{\pi} \sum_{r=1}^S I_r \int_0^{\infty} \frac{\mu_r \cos[T(y-\xi_r)] e^{-T(\eta_r+z)}}{\mu_r T + \sqrt{T^2 + j\omega\mu\sigma}} dT$$

where  $(\xi_r, \eta_r)$  are the coordinates of the  $r^{\text{th}}$  conductor and  $I_r$  is the current of the  $r^{\text{th}}$  conductor.

No series expansions for the integral were given, it being evaluated numerically.

(g) WEDEPOHL (1966)

The Multiple transform method is used to derive expressions for a transmission line with a two layer, horizontally stratified earth return in which both earth displacement currents and permeabilities are taken into account.

Result (M.K.S.A. units)

The impedance earth correction term for conductors r and s without displacement currents considered is:

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$$\Delta Z_{rs} = \frac{j\omega\mu_1}{\pi} \int_0^\infty \frac{\frac{\mu_2}{\mu_1} \cos[T(y-S_r)] e^{-T(h_r+z)}}{\frac{\mu_2}{\mu_1} T + A} dT$$

where  $(y, z)$  are the coordinates of conductor  $s$ ,  $(S_r, h_r)$  are the coordinates of conductor  $r$  and  $A$  is given by:

$$A = \frac{\alpha_2}{\sinh(d \cdot \alpha_2)} \left[ \cosh(d \cdot \alpha_2) - \frac{\mu_3 \alpha_2}{\mu_3 \alpha_2 \cosh(d \cdot \alpha_2) + \mu_2 \alpha_3 \sinh(d \cdot \alpha_3)} \right]$$

where  $\alpha_i = \sqrt{T^2 + m_i^2}$

and the parameters are defined according to Fig. (15.5).

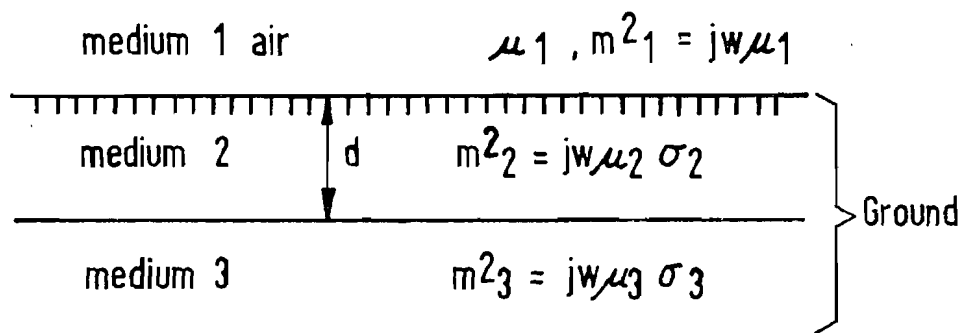


Fig. (15.5)

To include displacement currents  $A$  is modified thus:

$$\sqrt{T^2 + m_i^2} \quad \text{becomes} \quad \sqrt{T^2 + m_i^2 + K_i^2}$$

where  $K_i^2 = \omega^2 (\mu_1 \epsilon_1 - \mu_i \epsilon_i)$ .

#### Comment

For the homogeneous case  $A$  reduces to

$\sqrt{\alpha^2 + m_2^2 + \omega^2 (\mu_1 \epsilon_1 - \mu_2 \epsilon_2)}$  which is equivalent to the modification provided by WISE (1934) when  $\mu_1 = \mu_2 = \mu_0$ . As with WISE'S solution, the formula to account for displacement currents is

approximate, being derived on the assumption that the velocity of propagation is that of free space (efficient transmission). Note that if  $\mu_1 \epsilon_1 = \mu_i \epsilon_i$  no correction is necessary for displacement currents.

(h) SUNDE (1968)

SUNDE uses the Current Element Method to formulate the effect of the earth's finite conductivity for most problems in power and communications systems that are likely to be influenced by it. The results quoted here will be only for infinite overhead lines with earth returns.

Results (M.K.S.A. units)

For a uniform earth SUNDE obtains the mutual impedance:

$$(L = \frac{Z}{j\omega})$$

$$L = \frac{\mu}{2\pi} \left[ \log \frac{r'_{12}}{r_{12}} + W(\gamma r'_{12}) \right]$$

$$W(\gamma r'_{12}) = 2 \int_0^{\infty} \frac{\cos(Ty_{12}) e^{-T(h_1+h_2)}}{T + \sqrt{T^2 + \gamma^2}} \cdot dT$$

where  $\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon)$

and  $y_{12}$  = horizontal separation of conductors 1 and 2.

Expanding using CARSON'S series expansion for  $\gamma r'_{12} \ll \frac{1}{2}$  gives:

$$L = \frac{\mu}{2\pi} \left[ \log \frac{2}{\gamma r'_{12}} + \frac{1}{2} + \psi(0) + \frac{2}{3}\gamma(h_1+h_2) \right]$$

where  $\psi(0) = -0.5772.. = \log 1.7811.. = \text{Euler's constant.}$

SUNDE notes of this formula that in its range of application the last term <8% of the total, and for wire heights

actually encountered it will be much smaller than 8% and can be ignored. As an example of its range of application, for a frequency of 60 Hz, the above formula applies for separations up to approximately 100 metres when the earth resistivity is 100 ohm metres, and up to 1000 metres when the resistivity is 10,000 ohm metres.

For large separations, when  $\gamma r'_{12} \gg 5$  the following asymptotic expansion is obtained:

$$W(\gamma r'_{12}) = 2 \frac{\cos \theta}{\gamma r'_{12}} - 2 \frac{\cos 2\theta}{(\gamma r'_{12})^2} + 2 \frac{\cos 3\theta}{(\gamma r'_{12})^3}$$

$$\text{where } \tan \theta = \frac{y_{12}}{h_1 + h_2}$$

In the range where this applies  $\theta \approx 90^\circ$  in practical applications and:

$$W(\gamma r'_{12}) = \frac{2}{(\gamma r'_{12})^2}$$

For a conductor underneath and parallel to a lightning channel  $\gamma r'_{12}$  will usually exceed 5 in the frequency range of importance and  $\theta \approx 0$  making the first term dominant giving:

$$L = \frac{\mu}{\pi} \frac{h}{(h^2 + y^2) \gamma}$$

where  $h$  is the height of the lightning channel and  $y$  the horizontal separation.

For conductors with large separation over a two layer earth SUNDE obtains:

$$L = \frac{\mu}{\pi} \left[ \frac{\cos \theta}{\gamma_1' r_{12}'} - \frac{\cos 2\theta}{(\gamma_1' r_{12}')^2} \right]$$

$$\text{where } \gamma_1' = \gamma_1 \frac{1 - \eta e^{-2d\gamma_1}}{1 + \eta e^{-2d\gamma_1}}, \quad \eta = \frac{\sqrt{\rho_2} - \sqrt{\rho_1}}{\sqrt{\rho_2} + \sqrt{\rho_1}}$$

and subscripts 1 and 2 in the constants refer to the upper layer and lower layer of the ground with  $d$  the depth of the upper layer.

#### Comment

The above results are only representative of SUNDE'S work. Many other examples are given in his book which should be consulted by any person who is seriously working in this area.

#### (i) KAIDANOV (1965)

Carries out a rigorous analysis for the case of a fully symmetric mutually coupled two circuit transmission line with a uniform earth return. Displacement currents are taken into account and by not ignoring the transverse components of electric field intensity in the ground the wave nature of the field is accounted for. SOMERFELD'S solution for the field of a single conductor and superposition are used to determine expressions for the propagation constants of the two modes of propagation.

#### Results

By its very nature this paper is concerned with frequency ranges considerably higher than are likely to be met in power systems. The results obtained confirm that the assumptions usually made in formulating earth effects incur only small errors in the situations where the resulting formulas are likely to be used.

### 15.3.2) Steady State Sinusoidal Signals (Shunt Admittance)

#### (a) WISE (1948)

This paper uses the current element method to determine Maxwell's potential coefficients for a transmission line with a uniform homogeneous earth return. (The admittance coefficients of a line are found as the inverse of the potential coefficients.) The dipole formula and the assumptions used are the same as used in WISE (1934) with the propagation constant of free space being used as a satisfactory first approximation in evaluating correction terms. SUNDE (1968) observes that this is permissible for small conductor heights and typical earth resistivities.

#### Results (Electromagnetic c.g.s. units)

The potential coefficient between wires 1 and 2 is

$$p_{12} = c^2 \left[ 2 \log \frac{r'_{12}}{r_{12}} + 4(M+jN) \right]$$

where

$$M+jN = \int_0^\infty \frac{\cos(y\xi\sqrt{\alpha}t) e^{-(h+z)\xi\sqrt{\alpha}t}}{\sqrt{t^2+je^{j2\eta}} + (\epsilon-j2c\lambda\sigma)t} dt$$

$$\alpha = 4\pi\sigma\omega$$

$$\xi e^{j\eta} = \sqrt{1+j(\epsilon-1)/2c\lambda\sigma}$$

$$\xi = \sqrt{j}$$

It will be seen that  $M+jN$  vanishes as  $f \rightarrow 0$ ,  $f \rightarrow \infty$ ,  $\epsilon \rightarrow \infty$  or  $\sigma \rightarrow \infty$ .

WISE notes that ordinarily  $4(M+jN)$  will not be an important correction to  $2 \log r'_{12}/r_{12}$ ; but if the frequency

is high and  $h$  or  $z$  is small it can be a worthwhile correction. Approximate formulas for  $M+jN$  as an alternative to numerical integration are given where WISE comments that ordinarily precision is not required as the correction is small.

e.g. for wires vertically displaced only from each other

$$M+jN \approx -\frac{1}{a} e^{\frac{gu}{a}} \cdot \text{Li}(e^{-\frac{gu}{a}})$$

$$\text{where } \text{Li}(e^{-z}) = -\int_z^\infty \frac{e^{-t}}{t} dt = c + \log z + \sum_{\ell=1}^{\infty} \frac{(-z)^\ell}{\ell! \ell}$$

$$c = 0.577215665$$

$$g = (h+z) \xi \sqrt{\alpha}$$

$$u = e^{j(n + \frac{\pi}{4})}$$

$$a = \epsilon - j c \lambda \sigma$$

Curves given by WISE of  $M$  and  $N$  for wires with vertical separation only, show that the corrections become most significant when  $\frac{c\lambda\sigma}{9} = \frac{1}{4}$  for  $M$  and when  $\frac{c\lambda\sigma}{9} = 1.5$  for  $N$ . The correction for  $M$  falls off rapidly on both sides of its maximum and the correction for  $N$  falls off rapidly below its maximum value.

(b) ARISMUNANDAR (1963)

ARISMUNANDAR uses the Current Element Method to solve the same problem as WISE (1948). The formulation approach used follows closely that used by SUNDE (1968). The assumptions made are the same as those made by WISE (1948).

### Results (M.K.S.A. units)

For a single overhead line with an earth return

$$Y = \left[ \frac{1}{j\omega 2\pi\epsilon_0} \log \frac{2h}{a} + \frac{1}{\pi(\sigma + j\omega\epsilon)} \log \left( \gamma_0 h \left( 1 + \frac{\Delta\gamma_0}{\gamma_0} \right) \right) \right]^{-1}$$

where

$$\frac{\Delta\gamma_0}{\gamma_0} = \frac{\log F_p(2h, \gamma, \gamma_0)}{2 \log \left( \frac{2h}{a} \right)} - \frac{j\omega\epsilon_0}{\sigma + j\omega\epsilon} \frac{\log F_q(\gamma_0 h)}{\log \frac{2h}{a}}$$

$$\log F_p(\gamma, \gamma, \Gamma) = \frac{2}{\gamma\Gamma} \left[ \Gamma K_1(\Gamma\gamma) - \sqrt{\gamma^2 + \Gamma^2} K_1(\gamma\sqrt{\gamma^2 + \Gamma^2}) \right]$$

$$\log F_q(\Gamma\gamma) = K_0(\Gamma\gamma)$$

$$\gamma = j\omega\mu_0(\sigma + j\omega\epsilon)$$

By considering only the principal values of the logs and by separating real and imaginary parts the two quantities  $G$  (unit length conductance) and  $C_c$  (corrected capacitance) to ground are extracted.

ARISMUNANDAR quotes an example showing that application of the formulas for a typical switching surge frequency of 3 KHz and  $\sigma_e = 10^{-2}$  mhos/metre reveals the corrections for earth conductivity to be negligible. He concludes that no corrections need be made on the potential coefficient matrices for power system switching surge frequencies.

### (c) SUNDE (1968)

SUNDE uses the Current Element Method to derive the admittance of a transmission line with an earth return. A number of approximations are given. The result quoted is for a second approximation in terms of an exponential made of propagation and a uniform earth with  $\mu_r = 1$ .

Result (M.K.S.A. units)

$$Y = \left( \frac{1}{2\pi j\omega\epsilon_0} \log \frac{2h}{a} + \frac{1}{\pi(\sigma + j\omega\epsilon)} K_0(\Gamma a) \right)^{-1}$$

For  $\Gamma a < 0.01$  say,  $K_0(\Gamma a) \approx \frac{\log 1.12}{\Gamma a}$  can be substituted into the above.

$$\Gamma = \sqrt{Z(\beta)Y(\beta)}$$

where  $\beta$  is a constant chosen to give a good approximation to a logarithmically varying function (see SUNDE (1968), CHPT 5, Section 5.4). ARISMUNANDAR (1963) uses the propagation constant of free space, i.e.  $\Gamma = \gamma_0 = j\omega\sqrt{\mu_0\epsilon_0}$ .

Comment

SUNDE notes that this approximation yields substantially the same result as that obtained by WISE (1948) for small heights and typical resistivities but that it is more accurate than WISE'S result when the resistivity is high.

15.3.3) Operational Expressions (Series Impedance)

These expressions are derived to enable the impulse or step response of a line with finite conductivity to be found by inverse transformation of terms involving  $e^{-\gamma x}$ . In general only corrections to the series impedance and its effect on the surge impedance and propagation constant are considered. SUNDE (1968) shows that for aerial conductors, the effect of finite conductivity of the earth on the surge impedance is not particularly great for times of less than 10 microseconds after the wave has passed. Thus with sufficient accuracy for engineering purposes the surge impedance of a perfectly conducting earth can be used. Consequently, we will here consider only the propagation constant of the line.



This is often expressed in the form:

$$\gamma(s) = \sqrt{(sC+G)(sL+R+K\sqrt{s})}$$

It is frequently expanded binomially and set equal to the first two terms of the resulting series. Coefficient K embodies the correction for finite conductivity. It is usually divided into two components;

$$K = \xi + \beta$$

where  $\xi$  is a coefficient to correct for skin effect in the wires and  $\beta$  corrects for the presence of a semi-infinite finitely conducting earth return. We will mostly be concerned with coefficient  $\beta$ . Coefficient  $\xi$  will not be considered in detail although its derivation is treated in several of the references listed. In dealing with the propagation constant at high frequencies it is in fact permissible for typical conductor heights, radii and resistivities to neglect  $\xi$  (the internal impedance coefficient of the wires) in comparison with  $\beta$  (SUNDE, 1968).

The operational expressions are usually obtained by replacing in the impedance formulas already derived, the imaginary angular frequency ' $j\omega$ ' by the complex operators ' $p$ ' or ' $s$ '. e.g. RUDENBERG'S analysis for high frequencies (section 13.3.1)(e)) yields  $\beta = \sqrt{2\rho_g}/h$ .

(a) SHAH (1971)

SHAH derives the propagation constant for a single overhead conductor with an earth return. He uses CARSON'S expression (CARSON, 1926) for the electric field intensity

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along the surface of the ground to derive the high frequency internal impedance of the earth return. This he does in effect by reducing the infinite integral correction to the first term of its asymptotic expansion. The coefficient  $\beta$  is wholly composed of this, giving:

$$\beta = \frac{2}{h} \sqrt{\frac{\rho g}{\pi}}$$

The internal impedance of the wire is given by a formula due to MINE (1952):

$$\alpha = \frac{1}{a} \sqrt{\frac{\mu_c \rho_c}{\pi}}$$

By including no further correction terms SHAH's formulation takes no account of the effect that the distribution of ground currents has on the magnetic field in air.

(b) ROBERT and TRAN (1969, 1972, 1972)

ROBERT and TRAN in a series of papers derive operational expressions for the series impedances and propagation constants associated with both single and multiple circuit transmission lines. The effect of the earth is taken into account using an operational form of CARSON'S correction. Skin effect in the wires is taken into account by solving the wave equation for the conductor with cylindrical symmetry. The resulting propagation constant is defined for exponential propagation by  $e^{-\frac{\gamma x}{c}}$  where  $\gamma^2 = p(p + a_c + a_g)$ . The terms  $a_c$  and  $a_g$  account for the finite conductivity of the wires and ground respectively. For the wires:

$$a_c = \frac{C\rho_c}{\mu_0\epsilon_0 2\pi r_c} z_1 \frac{I_0(z_1)}{I_1(z_1)}$$

where  $z_1 = a_1 r_c$

$$a_1^2 = \mu_0(\sigma_c p + \epsilon_0 p^2)$$

$r_c$  = radius of conductor.

For the earth ROBERT uses:

$$\begin{aligned} a_g &= \frac{Cp}{\pi\epsilon_0} \int_0^\infty \frac{e^{-2h\alpha}}{\sqrt{a^2 + \alpha^2} + \alpha} d\alpha \\ &= \frac{Cp}{\pi\epsilon_0} \left\{ \frac{\pi}{2z_2} [H_1(z_2) - Y_1(z_2)] - \frac{1}{z_2^2} \right\} \end{aligned}$$

where  $z_2 = 2ha_2$

and  $a_2^2 = \mu_0(\sigma_g p + \epsilon_0 p^2)$

To simplify the expressions for the propagation constant, approximations are made for large and small values of  $p$  which give responses valid for short and long periods of time after the wave has passed. The displacement currents in the conductors and ground are neglected and  $\gamma$  is approximated by the first two terms of its binomial expansion.

By considering large values of 'p' the propagation constant for small values of time becomes:

$$\gamma \approx p + \beta_0 \sqrt{p} + \alpha_0$$

$$\text{where } \beta_0 = \frac{C}{2\pi\epsilon_0\sqrt{\mu_0}} \left[ \frac{\sqrt{\rho_c}}{2r_c} + \frac{\sqrt{\rho_g}}{2h} \right]$$

$$\text{and } \alpha_0 = \frac{C}{2\pi\epsilon_0\mu_0} \left[ \frac{\rho_c}{4r_c^2} - \frac{\rho_g}{4h^2} \right]$$

For larger values of time the two terms  $a_c$  and  $a_g$  are developed differently. On account of the widely different values of  $\sigma_c$  and  $\sigma_g$  there is an interval of  $p$  where  $a_c$  can be developed for large values of  $z_1$  and  $a_g$  can be developed for small values of  $z_2$ . (i.e. to give a formula valid over an interval of time, e.g. 150-1500 microseconds say). When  $z_2$  is small

$$a_g = \frac{Cp}{\pi\epsilon_0} \left( \frac{1}{2} \log \frac{2}{mg\sqrt{p}} - 0.0386 \right)$$

where  $m_g = 2h\sqrt{\mu_0\sigma_0}$ . For  $z_2$  small, the further approximation of the log by a straight line is made;

$$\log \frac{2}{m_g\sqrt{p}} \approx A_1 + A_2 \cdot \frac{2}{m_g\sqrt{p}}$$

where  $A_1$  and  $A_2$  are chosen to give small errors over the interval of  $z_2$  being considered. The propagation constant is then derived as:

$$\gamma = p(1+\tau_m) + \beta_f\sqrt{p} + \alpha_f$$

$$\text{where } \tau_m = \frac{C}{\pi\epsilon_0} \left( \frac{A_1}{4} - \frac{0.0368}{2} \right)$$

$$\beta_f = \frac{C}{2\pi\epsilon_0\sqrt{\mu_0}} \left[ \frac{\sqrt{\rho_c}}{2r_c} + A_2 \frac{\sqrt{\rho_g}}{2h} \right]$$

$$\alpha_f = \frac{C\rho_c}{8\pi\epsilon_0\mu_0r_c}$$

#### Comment

As pointed out by the authors expanding  $a_c$  and  $a_g$  differently is, surprisingly, a new feature of this work. The constant  $\tau_m$  represents a mean percentage additional delay in

the arrival of the wave. It measures in percentage the excess time delay in the presence of finite conductivity over the velocity of light. By letting the interval tend to zero and using the tangent to the log curve an instantaneous delay term  $\tau_i(t)$  can be obtained which allows a response to be developed from time zero up to say 1500 microseconds through the intermediate range of times that are not ordinarily available in a simple closed form expression. This constitutes a further new feature of the work carried out in this reference. The resulting response is considered in section 16.2.4).

(c) SUNDE (1968)

SUNDE uses the correction terms he has derived for the series impedance to construct the propagation constant from:

$$\Gamma = \sqrt{pC(Z_i + pL)}$$

where for a single overhead conductor and a uniform earth

$Z_i = \frac{1}{2\pi a} \sqrt{p\mu\rho_c}$  is the internal impedance of the conductor at high frequencies, and  $L = \frac{\mu}{2\pi} [\log \frac{2h}{a} + W(\gamma h)]$  (section 15.3.1(h)).

With sufficient accuracy for engineering purposes SUNDE approximates  $W$  by:

$$W = \log \frac{1+\gamma h}{\gamma h}$$

This gives the propagation constant  $\Gamma = \frac{p}{c} H(p)$  which when approximated by the first two terms of its binomial expansion gives:

$$H(p) \approx 1 + \frac{1}{2} \left( \log \frac{2h}{a} \right)^{-1} \left( \log \frac{1+K\sqrt{p}}{K\sqrt{p}} + \frac{1}{K_1\sqrt{p}} \right)$$

$$\text{where } K = h\sqrt{\mu\sigma_g}, \quad K_1 = a\sqrt{\mu\sigma_c}$$

The propagation constant is then expanded for large and small values of  $p$ . For large  $p$ ;

$$\Gamma = \frac{p}{c} \left[ 1 + (2K_1\sqrt{p} \log \frac{2h}{a})^{-1} \right]$$

$$\text{where } K_1 = \frac{K \cdot K_i}{K + K_i}$$

For small values of  $p$ ,  $1+K\sqrt{p} \approx 1$  in  $H(p)$ .

(d) MAGNUSSON

MAGNUSSON has solved, and has contributed to the correspondence of, travelling wave problems on transmission systems with finite conductivity (MAGNUSSON 1968, 1972, (discussion SHAH, 1971)).

In the discussion on SHAH (1971) MAGNUSSON takes as a high frequency approximation to CARSON'S (1926) correction the first term of its asymptotic expansion giving:

$$\beta = \frac{1}{h} \sqrt{\frac{\rho_g}{\pi}}$$

This correction coefficient, unlike SHAH'S, includes the effect of the ground current distribution on the magnetic field in air (cf. SHAH (1971),  $\beta = \frac{2}{h} \sqrt{\frac{\rho_g}{\pi}}$ ).

In MAGNUSSON (1972) asymptotic impedance approximations were examined for a fully symmetric two circuit transmission line. Here the series impedance was approximated by the expansion:

$$Z(s) = sL + a_1\sqrt{s} + a_2 + \frac{a_3}{\sqrt{s} + a_4}$$

The constant  $a_4$  is added to make the impedance finite at zero frequencies. The constants can be chosen to give the true asymptotic expansion or can be chosen to fit the exact formulation at four discrete points. (In this reference CARSON'S (1926) results were used as the exact solution.) This is then used in the propagation constant,  $\gamma = \sqrt{sC \cdot Z(s)}$ , in which the terms are grouped into useful combinations and once again the binomial expansion is used. An intuitive empirical approximation for the series impedance was also examined.

#### Comment

The resulting series solutions were shown to have slow convergence throughout much of the time span they involved.

#### (e) General Comment

It will have been noticed that SHAH, TRAN and MAGNUSSON have all taken a high frequency approximation to a result that is valid only at low frequencies (a point not made by these authors). Hence, strictly speaking, their resulting impulse and step responses cannot be used for travelling waves or surges over short periods of time. They will possibly be useful in a convolution solution of slower transient problems as may be found in power transmission systems. In real terms, however, the resulting solution errors will not be as significant as may at first appear since although CARSON'S (1926) correction terms can be in

error by as much as 80% at frequencies of the order of 5 MHz (KAIDANOV, 1965), the magnitudes of the correction terms decrease as the frequency rises.

#### 15.3.4) Time Formulation (Series impedance)

The only time domain formulation of the effects of finite conductivity encountered by the author is that carried out by MILLER (1947) although a time domain equivalent of MULLINEUX'S (1965) result is derived in section 15.4) of this thesis. By neglecting displacement currents, MILLER'S results are strictly applicable only to good conductors (section 15.2.1)). His results have, however, been applied to a finitely conducting ground (WAGNER, 1963) and since his work gives a good physical insight into the effects of finite conductivity it is considered here.

##### (a) MILLER (1947)

MILLER solves for the distribution of current density across a conductor in response to an applied step of current. He uses the Diffusion Equation Method and his solution is in the time domain.

##### Results (M.K.S.A. units)

For parallel transverse flow into a semi-infinite plate the distribution of current density is:

$$i(x,t) = \frac{I}{h\sqrt{\pi t}} e^{-\frac{x^2}{4h^2 t}}$$

where I is the magnitude of the current and x is the depth being considered. Such a distribution can be considered a



suitable approximation for round conductors only for a very short time after the current step is applied (high frequencies). For this latter case  $x$  would be the radial depth from the surface.

The solution for a step of current applied to a solid round conductor is:

$$i(x,t) = \frac{I}{\pi b^2} \left[ 1 + \sum_{n=1}^{\infty} \frac{J_0(u_n x)}{J_0(u_n b)} e^{-h^2 u_n^2 t} \right]$$

where  $J_1(u_n b) = 0$  and  $x$  is the radial depth from the surface. This leads to a transient internal inductance and resistance:

$$L = \mu \left[ \frac{1}{2} - 4t \sum_{n=1}^{\infty} (e^{-u_n^2 t}) (2 - e^{-u_n^2 t}) / (u_n^2 t) \right]$$

$$R = 2I^2 \sqrt{\frac{\mu \rho}{2t}}$$

#### Comment

MILLER points out that the transient current density for any current wave can be determined by utilizing the Duhamel integral.

This reference shows skin effect to be a sidewise penetration of current density into a conductor in a direction normal to the lines of electric current flow. The diffusion velocity of this penetration is  $h/\sqrt{t}$  where  $\frac{4\pi\mu}{\rho} = \frac{1}{h^2}$ . The two dimension treatment of this problem is justified by the propagation velocity of the current wave ( $1/\sqrt{\mu_0 \epsilon_0}$ ) vastly exceeding the current density diffusion velocity (e.g. for copper  $h = 11.6 \text{ cm}/\sqrt{\text{sec.}}$ ).

#### 15.4) DETAILED FORMULATION OF THE T.L.E.s WITH A FINITELY CONDUCTING EARTH

The basis of this formulation will be the multiple transform analysis of MULLINEUX (1965). The analysis given here, however, will differ in that it is carried through to the time domain. The analysis is for a multiple circuit transmission line above a uniform earth where the permeability of the earth is taken into account.

In this analysis the wave equation will be written in the air and in the ground. These equations will be transformed with respect to their coordinates and matched to the boundary conditions along the earth's surface. On taking inverse transforms an expression is obtained for the electric field intensity in air due to the conductor currents and their resulting distribution of ground currents.

The coordinate system used by MULLINEUX will be used here. It is different from that used previously in this section and is shown in Fig. (15.6).

##### 15.4.1) Wave Equation and Assumptions

The wave equation for a conducting medium is:

$$\nabla^2 E - \mu \epsilon \frac{\partial^2 E}{\partial t^2} = \mu \sigma \frac{\partial E}{\partial t}$$

Taking complex fourier transforms w.r.t. time subject to the conditions in appendix (15.1.1) gives:

$$\nabla^2 E + \omega^2 \mu \epsilon E = j \omega \mu \sigma E$$

The following assumptions are implicit in MULLINEUX (1965) but have not been stated.

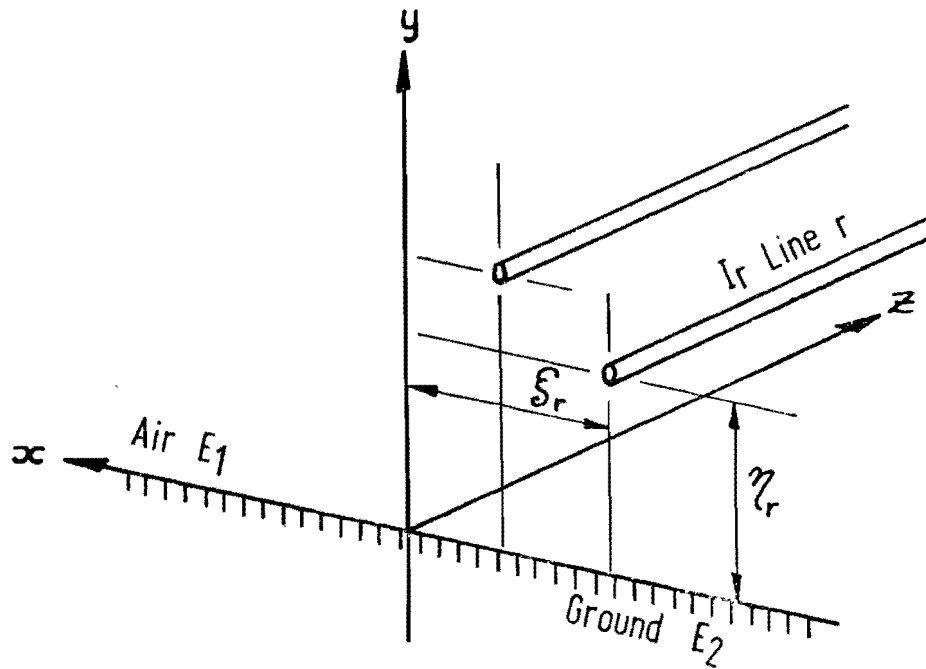


Fig. (15.6) Coordinate system for multiple transform analysis

(a) Wave Equation in Air

- 1) The wave is propagating exponentially. Separating out the  $z$  component from the Laplacian gives:

$$\nabla_{xy}^2 E_1 + (\gamma^2 + \omega^2 \mu_0 \epsilon) E = j\omega \mu_0 \sigma E$$

- 2) The wave is propagating at the speed of light and without attenuation. i.e.  $(\gamma^2 + \omega^2 \mu_0 \epsilon) = 0$ .
- 3) The current is concentrated in infinitely thin cylinders along the axis of the conductor so that

$$J = \sigma E = \sum_{r=1}^S I_r \delta(x - \xi_r) \delta(y - \eta_r)$$

where  $I_r$  is the total current of the  $r^{\text{th}}$  conductor which has coordinates  $(\xi_r, \eta_r)$ .

(b) Wave Equation in Ground

- 4) Displacement currents are considered to be negligibly small compared with conduction currents. i.e.  $\frac{\omega \epsilon}{\sigma} \ll 1$ .
- 5) Transverse electric field components are negligible compared with the axial components. i.e.  $E_x = E_y = 0$ ,  $E = E_z$ .
- 6) Axial ground currents are considered constant. i.e. The series impedance is being calculated separately from the shunt admittance.

(c) Comment

It should be noted of MULLINEUX'S paper that his statement accusing CARSON'S (1926) formulation of presupposing images is incorrect. The log term is a natural consequence of  $\int_0^y H(x,y) dy$ . (Equation (17) in CARSON'S paper) This is amply demonstrated in BROWN (1971) where the integration is carried out.

15.4.2) Multiple Transform Analysis(a) Solving for E Transformed in Air and Ground

For  $y < 0$  the variable  $\psi = -y$  has been used. Considering a multiconductor line with 's' conductors we have:

$$y > 0 \quad \frac{\partial^2 E_1}{\partial x^2} + \frac{\partial^2 E_1}{\partial y^2} = m_1^2 \sum_{r=1}^s I_r \delta(x - \xi_r) \delta(y - \eta_r)$$

$$y < 0 \quad \frac{\partial^2 E_2}{\partial x^2} + \frac{\partial^2 E_2}{\partial \psi^2} = m_2^2 E_2$$

$$\text{where } m_1^2 = j\omega\mu_0 \quad \text{and} \quad m_2^2 = j\omega\mu_r\mu_0\sigma.$$

Fourier sine transforms (transform parameter  $\alpha$ ) are taken w.r.t.  $y$  and  $\psi$  (APPENDIX (15.1.2)). (This transform is chosen because the range of integration matches the semi infinite variables  $y$  and  $\psi$  and because it gives a single function initial condition to the second derivative simplifying matching to the physical situation.)

$$\frac{d^2 E_1^*}{dx^2} - \alpha^2 E_1^* = \alpha \sqrt{\frac{2}{\pi}} E_{10} + m_1^2 \sum_{r=1}^S I_r \delta(x - \xi_r) \sqrt{\frac{2}{\pi}} \sin \alpha \eta_r$$

$$\frac{d^2 E_2^*}{dx^2} - \alpha^2 E_2^* = \alpha \sqrt{\frac{2}{\pi}} E_{20} + m_2^2 E_2^*$$

$$\text{where } E^* = \sqrt{\frac{2}{\pi}} \int_0^\infty E \sin(\alpha y) dy$$

Taking complex fourier transforms w.r.t.  $x$ , transform parameter  $\beta$ , gives: (APPENDIX (15.1.3))

$$-\beta^2 E_1^{**} - \alpha^2 E_1^{**} = \alpha \sqrt{\frac{2}{\pi}} E_{10}^* + m_1^2 \sum_{r=1}^S I_r e^{-j\beta \xi_r} \sqrt{\frac{2}{\pi}} \sin \alpha \eta_r$$

$$-\beta^2 E_2^{**} - \alpha^2 E_2^{**} = \alpha \sqrt{\frac{2}{\pi}} E_{20}^* + m_2^2 E_2^{**}$$

$$\text{where } E^{**} = \int_{-\infty}^\infty E^* e^{-j\beta x} dx$$

Solving for  $E_1^{**}$  and  $E_2^{**}$  gives:

$$E_1^{**} = -\frac{\alpha \sqrt{\frac{2}{\pi}}}{\alpha^2 + \beta^2} E_{10}^* - m_1^2 \sum_{r=1}^S I_r \frac{e^{-j\beta \xi_r} \sin \alpha \eta_r}{\alpha^2 + \beta^2}$$

$$E_2^{**} = -\frac{\alpha \sqrt{\frac{2}{\pi}}}{\alpha^2 + \beta^2 + m_2^2} E_{20}^*$$

Taking inverse sine transforms w.r.t.  $y$  and  $\psi$  from  $\alpha$  gives  
(APPENDIX (15.1.4))

$$E_1^{*Y} = -E_{10}^* e^{-|\beta y|} - m_1^2 \sum_{r=1}^s I_r e^{-j\beta \xi_r} \\ \cdot \frac{1}{2|\beta|} \left[ e^{-|\beta(\eta_r - y)|} - e^{-|\beta(\eta_r + y)|} \right]$$

$$E_2^{*\psi} = -E_{20}^* e^{-|\sqrt{\beta^2 + m_2^2} \cdot \psi|}$$

$$\text{where } E^{*Y} = \sqrt{\frac{2}{\pi}} \int_0^\infty E^{**} \sin(\alpha y) d\alpha$$

(b) System Boundary Conditions

The boundary conditions at  $y = \psi = 0$  are:

- (1)  $E$  is continuous, i.e.  $E_{10} = E_{20}$
- (2)  $H_x$  is continuous, i.e.  $\frac{\partial E_1}{\partial y} = \frac{1}{\mu_r} \frac{\partial E_2}{\partial y}$
- (3)  $B_y$  is continuous, i.e.  $\frac{\partial E_1}{\partial x} = \frac{\partial E_2}{\partial x}$

(c) Matching Equations at Boundary and Solving for  $E_1$

Differentiating w.r.t.  $y$  gives (APPENDIX 15.2)):

$$\frac{\partial E_1^{*Y}}{\partial y} = |\beta| E_{10}^* e^{-|\beta y|} - m_1^2 \sum_{r=1}^s I_r e^{-j\beta \xi_r} \\ \cdot \frac{1}{2} \left[ e^{-|\beta(\eta_r - y)|} + e^{-|\beta(\eta_r + y)|} \right] \quad (15.6)$$

Differentiating w.r.t.  $\psi$  gives (APPENDIX 15.2)):

$$\frac{\partial E_2^{*\psi}}{\partial \psi} = \sqrt{\beta^2 + m_2^2} E_{20}^* e^{-|\sqrt{\beta^2 + m_2^2} \cdot \psi|} \quad (15.6a)$$

Applying boundary condition (2) noting that  $\frac{\partial}{\partial \psi} = -\frac{\partial}{\partial y}$ :

$$|\beta| E_{10}^* - m_1^2 \sum_{r=1}^s I_r e^{-j\beta \xi_r} e^{-|\beta \eta_r|} = -\frac{1}{\mu_r} \sqrt{\beta^2 + m_2^2} E_{20}^*$$

Using boundary condition (1) we obtain:

$$E_{10}^* = \frac{m_1^2 \sum_{r=1}^s I_r e^{-j\beta \xi_r} e^{-|\beta \eta_r|}}{|\beta| + \frac{1}{\mu_r} \sqrt{\beta^2 + m_2^2}}$$

Substituting for  $E_{10}^*$  in equation (15.6) an expression for  $E_1^{*y}$  can be obtained:

$$\begin{aligned} E_1^{*y} &= -m_1^2 \sum_{r=1}^s I_r \frac{e^{-j\beta \xi_r} e^{-|\beta(\eta_r+y)|}}{|\beta| + \frac{1}{\mu_r} \sqrt{\beta^2 + m_2^2}} \\ &\quad - m_1^2 \sum_{r=1}^s I_r e^{-j\beta \xi_r} \left[ \frac{e^{-|\beta(\eta_r-y)|} - e^{-|\beta(\eta_r+y)|}}{2|\beta|} \right] \end{aligned}$$

Taking inverse complex fourier transforms w.r.t.  $x$  from  $\beta$  gives (APPENDIX (15.1.5)):

$$\begin{aligned} E_1 &= \frac{m_1^2}{4\pi} \sum_{r=1}^s I_r \log \left[ \frac{(\eta_r - y)^2 + (x - \xi_r)^2}{(\eta_r + y)^2 + (x - \xi_r)^2} \right] \\ &\quad - \frac{m_1^2}{\pi} \sum_{r=1}^s I_r \int_0^\infty \frac{\cos \beta(x - \xi_r) e^{-\beta(\eta_r + y)}}{\beta + \frac{1}{\mu_r} \sqrt{\beta^2 + m_2^2}} d\beta \end{aligned} \quad (15.7)$$

This gives the axial electric field intensity in the air as a function of the system geometry, the magnitude of the line currents and their frequency. The second term gives the effect of the finitely conducting ground. It tends to zero as  $\sigma$  tends to infinity. CARSON'S (1926) expression can be obtained by setting the relative permeability of the ground ' $\mu_r$ ' to unity. The effect of displacement currents in the ground can be included simply by setting  $m_2 = j\omega\mu(\sigma + j\omega\epsilon)$ .

To obtain an expression for E in the time domain it is necessary to carry out a further transformation from the frequency domain to the time domain. Doing this the first term becomes:

$$\frac{\mu_0}{4\pi} \sum_{r=1}^S \left( \frac{\partial}{\partial t} I_r \right) \log \left[ \frac{(\eta_r - y)^2 + (x - \xi_r)^2}{(\eta_r + y)^2 + (x - \xi_r)^2} \right]$$

The second term of equation (15.7) can be written:

$$- \frac{\mu_r \mu_0}{\pi} \sum_{r=1}^S \int_0^{\infty} \frac{j\omega I_r}{\mu\beta + \sqrt{\beta^2 + j\omega\mu_0\mu_r\sigma}} \cos \beta(x - \xi_r) e^{-(\eta_r + y)\beta} d\beta$$

and has the frequency dependence:

$$\frac{j\omega I_r}{\mu_r\beta + \sqrt{\beta^2 + j\omega\mu_0\mu_r\sigma}} = f(\omega) \cdot h(\omega)$$

Now

$$F_T^{-1} [f(\omega) \cdot h(\omega)] = \int_{-\infty}^{\infty} g(\beta, T) \cdot \dot{I}(Z, t - T) dT$$

where



$$\begin{aligned}
 g(\beta, t) &= F_T^{-1} [h(\omega)] \\
 &= F_T^{-1} [1/(\mu_r \beta + \sqrt{\beta^2 + j\omega \mu_0 \mu_r \sigma})]
 \end{aligned}$$

It is shown in APPENDIX (15.1.6) that:

$$g(\beta, t) = \frac{e^{\frac{-t\beta^2}{\mu_r \mu_0 \sigma}}}{\sqrt{\pi \mu_r \mu_0 \sigma t}} - \frac{\beta}{\mu_0 \sigma} e^{\frac{t\beta^2 (\mu_r^2 - 1)}{\mu_r \mu_0 \sigma}} \operatorname{erfc} \left( \beta \sqrt{\frac{t \mu_r}{\mu_0 \sigma}} \right)$$

which is valid for  $t > 0$ ,  $\beta \geq 0$  and  $\sigma > 0$ .

Thus the electric field intensity in time, produced by a group of 's' horizontal parallel conductors above a finitely conducting earth is given by:

$$\begin{aligned}
 E_1 &= \frac{\mu_0}{4\pi} \sum_{r=1}^s \dot{I}_r (+) \log \left[ \frac{(\eta_r - y)^2 + (\xi_r - x)^2}{(\eta_r + y)^2 + (x - \xi_r)^2} \right] \\
 &\quad - \frac{\mu_r \mu_0}{\pi} \sum_{r=1}^s \left\{ \int_0^\infty \int_{0+}^\infty g(\beta, T) \dot{I}_r(z, t-T) dT \right\} \cos \beta(x - \xi_r) e^{-(\eta_r + y)\beta} d\beta
 \end{aligned}$$

#### 15.4.3) T.L.E.s Extended to Include the Effects of a Finitely Conducting Earth

The derivation of the extended T.L.E.s is carried out in an identical manner to the derivation given in section 3) but here the equation of voltage balance is (for a single overhead conductor with an earth return):

$$v(a,t) - v(a+\Delta z,t) = \int_a^{a+\Delta z} \{R \cdot i(z,t) + L \frac{\partial i}{\partial t}(z,t) + P(z,t)\} dz$$

$$\text{where } L = L_{\text{int}} + \frac{\mu_0}{2\pi} \log \frac{2h}{a}$$

$$\text{and } P(z,t) = \frac{\mu g}{\pi} \int_0^\infty \left\{ \int_{0+}^\infty g(\beta, T) \dot{i}(z, t-T) dT \right\} e^{-2h\beta} d\beta$$

This gives the extended T.L.E.s for a single circuit transmission line:

$$-\frac{\partial v}{\partial z}(z,t) = R \cdot i(z,t) + L \frac{\partial i}{\partial t}(z,t) + P(z,t)$$

$$-\frac{\partial i}{\partial z}(z,t) = G \cdot v(z,t) + C \frac{\partial v}{\partial t}(z,t)$$

The extended T.L.E.s for a mutually coupled multiple circuit transmissionline are derived similarly but now contain terms describing mutual effects. As with the case of a single circuit line, each inductive term has an associated earth modifying term.

Reversing the order of integration  $P(z,t)$  can be written:

$$P(z,t) = \frac{\mu g}{\pi} \int_{0+}^\infty \left\{ \int_0^\infty g(\beta, T) e^{-2h\beta} d\beta \right\} \dot{i}(z, t-T) dT$$

and carrying out the integration over  $\beta$  gives (APPENDIX (15.3)):

$$P(z,t) = \frac{\mu g}{\pi} \int_{0+}^\infty W(T) \dot{i}(z, t-T) dT$$

where

$$W(T) = \left\{ \frac{2}{\sqrt{\pi}} h \sqrt{\frac{\mu_0 \sigma}{T}} + e^{h^2 \frac{\mu_0 \sigma}{T}} \operatorname{erfc} \left( h \sqrt{\frac{\mu_0 \sigma}{T}} \right) - 1 \right\} / (2h \sqrt{\mu_0 \sigma})^2$$

in which  $\mu_r = 1$  and which is valid for  $T > 0$ .

#### Comment

The extended T.L.E.s are thus the T.L.E.s as for a perfectly conducting earth with an extra term,  $P(z,t)$ , to take into account the effects of the earth's finite conductivity.  $P(z,t) \rightarrow 0$  as  $\sigma \rightarrow \infty$ . The correction term, being derived from a low frequency approximation, is valid only for slow transients. It is seen that it modifies the T.L.E.s by including the effect of signals that have already passed. In terms of MILLER'S (1947) work, the transmission system geometry is altered as the current density diffuses from the surface of the ground downwards, effectively altering the physical dimensions of the earth conductor.

It is very difficult to utilize the extended T.L.E.s in this form in a time domain solution. The lower limit of integration is undefined and the time derivative of the line current could introduce computational errors. Applying the method of characteristics to the extended T.L.E.s as they stand necessitates integrating  $P(z,t)$  along a characteristic curve which, while it can be done with suitable approximations, is very cumbersome and presents considerable difficulties. The method of characteristics has been applied to this problem much more effectively by utilizing numerical inverse transformation into the time domain (SNELSON (1972), section 16.4.4) of this thesis).

### 15.5) IN CONCLUSION

Many different formations of the effect of the earth's finite conductivity on transmission systems have been carried out. All are approximate; some less so than others. Rigorous solutions to this problem would seldom be needed in engineering situations and could even more seldomly be justified in view of the probable uncertainty with which the basic earth parameters would be known. The original formulation by CARSON (1926) still appears to be the most commonly used formulation in power systems analyses although with digital computers the more complete formulation of WEDEPOHL (1966) could be used with little extra effort.

Almost all the formulations have been carried out in the frequency domain and can be incorporated readily into frequency domain solutions (section 16.1)). Solution in the time domain is more difficult and operational expressions have traditionally been derived for this purpose. (Although numerical inversion techniques have recently appeared, section 16.4).) In these formulations there seems to be a common tendency amongst power systems engineers to consider high frequency approximations to formulae which are valid only for low frequencies. This practice should be treated with caution as the resulting formulae are, strictly speaking, not valid for either high or low frequencies. Whether this approximation leads to large errors or whether it is insignificant compared with other approximations has not been investigated in this section although it is pointed out (section 15.3.3)(e)) that its effect could well be less than might at first be suspected.

At power system frequencies, usually only corrections to the transmission line's series impedances are significant. The usual form of an earth corrected formulation of the series impedance is for it to consist of the same terms as for a perfectly conducting earth plus a correction term which tends to zero as the earth's conductivity tends to infinity. The correction term is often expressed as an infinite integral although sometimes this integral can be expressed in terms of mathematical functions. In the author's experience these functions are not usually supported by current computer software. CARSON (1926) and others have given series expressions for their integrals and these are still frequently used although more recently numerical integration of the integrals has become common.

Of the assumptions commonly made in deriving the earth's effects, only two appear to be overpoweringly significant: (1) Neglecting displacement currents in the earth (restricts formulation to low frequencies), (2) assuming the earth to be uniform and homogeneous. Of these two, assuming the earth to be uniform and homogeneous appears to be most damaging. Only SUNDE (1968) and WEDEPOHL (1966) have given formulations that do not neglect displacement currents in the earth and which also consider the earth to be other than uniform and homogeneous.

## APPENDIX (15.1)

### DERIVATION OF TRANSFORMS FOR MULTIPLE TRANSFORM ANALYSIS

#### (15.1.1) Complex Fourier Transform

The complex Fourier transform is defined as:

$$F_T(f(t)) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

(a)  $f(t) = \dot{g}(t) = \frac{d}{dt} g(t)$

$$\begin{aligned} F_T(\dot{g}(t)) &= \int_{-\infty}^{\infty} \dot{g}(t) e^{-j\omega t} dt \\ &= [g(t) e^{-j\omega t}]_{-\infty}^{\infty} + j\omega \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt \end{aligned}$$

or  $F_T(\dot{g}(t)) = j\omega F_T(g(t))$

Provided that  $g(t) \rightarrow 0$  as  $t \rightarrow \pm\infty$

(b)  $f(t) = \ddot{g}(t) = \frac{d^2}{dt^2} g(t)$

$$F_T(\ddot{g}(t)) = \int_{-\infty}^{\infty} \ddot{g}(t) e^{-j\omega t} dt$$

Integrating twice by parts gives:

$$\underline{F_T(\ddot{g}(t)) = -\omega^2 F_T(g(t))}$$

provided that  $\dot{g}(t)$  and  $g(t) \rightarrow 0$  as  $t \rightarrow \pm\infty$

(15.1.2) Fourier Sine Transforms

The Fourier sine transform is defined as

$$S_T(f(y)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(y) \sin(\alpha y) dy$$

(a)  $f(y) = \delta(y - \eta_r)$

$$\begin{aligned} S_T(\delta(y - \eta_r)) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \delta(y - \eta_r) \sin(\alpha y) dy \\ &= \underline{\sqrt{\frac{2}{\pi}} \sin(\alpha \eta_r)} \end{aligned}$$

(b)  $f(y) = \ddot{g}(y)$

$$S_T(\ddot{g}(y)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \ddot{g}(y) \sin(\alpha y) dy$$

Integrating twice by parts gives:

$$\underline{S_T(\ddot{g}(y)) = -\alpha \sqrt{\frac{2}{\pi}} g(0) - \alpha^2 S_T(g(y))}$$

provided that  $\dot{g}(y)$  and  $g(y) \rightarrow 0$  as  $y \rightarrow \infty$

(15.1.3) Complex Fourier Transform

(a)  $f(x) = \ddot{g}(x)$

See part (15.1.1) of  
APPENDIX (15.1).

$$(b) \quad f(x) = \delta(x - \xi_r)$$

$$\begin{aligned} F_T(\delta(x - \xi_r)) &= \int_{-\infty}^{\infty} \delta(x - \xi_r) e^{-j\beta x} dx \\ &= \underline{e^{-j\beta \xi_r}} \end{aligned}$$

#### (15.1.4) Inverse Fourier Sine Transforms

The inverse Fourier sine transform is defined as:

$$S_T^{-1}(f(\alpha)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(\alpha) \sin(\alpha y) d\alpha$$

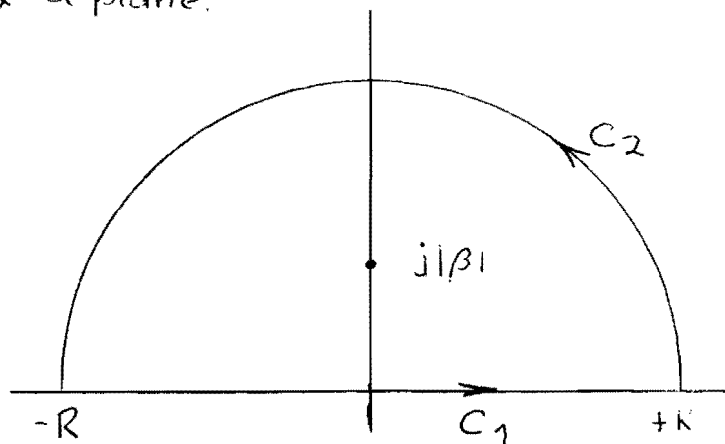
$$(a) \quad f(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 + \beta^2}$$

$$S_T^{-1}(f(\alpha)) = \frac{2}{\pi} \int_0^{\infty} \frac{\alpha}{\alpha^2 + \beta^2} \sin(\alpha y) d\alpha$$

which, since the integrand is even:

$$= \frac{1}{\pi} \oint_{\Gamma} \frac{\alpha}{\alpha^2 + \beta^2} e^{j\alpha y} d\alpha$$

This can be evaluated using contour integration in the complex  $\alpha$  plane.







$$(b) \quad f(\alpha) = \frac{\sqrt{\frac{2}{\pi}} \sin(\alpha \eta_r)}{\alpha^2 + \beta^2}$$

$$\begin{aligned} S_T^{-1}(f(\alpha)) &= \frac{2}{\pi} \int_0^{\infty} \frac{\sin(\alpha \eta_r) \sin(\alpha y)}{\alpha^2 + \beta^2} d\alpha \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos(\alpha |\eta_r - y|) - \cos(\alpha |\eta_r + y|)}{\alpha^2 + \beta^2} d\alpha \\ &= \frac{1}{2\pi} \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{j\alpha |\eta_r - y|} - e^{j\alpha |\eta_r + y|}}{\alpha^2 + \beta^2} d\alpha \end{aligned}$$

(Use of  $|\eta_r - y|$  and  $|\eta_r + y|$  is necessary to ensure that the integral around  $C_2 \rightarrow 0$ )

Considering the case for  $\beta \neq 0$ , integrate around the closed contour  $C = C_1 + C_2$  in the complex  $\alpha$  plane which encloses a simple single order pole at  $\alpha = j|\beta|$ .

$$\int_C = 2\pi j \times \{\text{Residue of integrand at } \alpha = j|\beta|\}$$

As  $R \rightarrow \infty$ , along  $C_2$

$$|\text{Integrand}| < \frac{|e^{-R \sin \theta |\eta_r - y|} - e^{-R \sin \theta |\eta_r + y|}|}{R}$$

and  $\int_{C_2} \rightarrow 0$  as  $R \rightarrow \infty$  provided that  $0 < \theta < \pi$ .

$$\text{Therefore } \frac{2}{\pi} \int_0^{\infty} \frac{\sin(\alpha \eta_r) \sin(\alpha y)}{\alpha^2 + \beta^2} d\alpha = \frac{[e^{-|\beta|(\eta_r - y)} - e^{-|\beta|(\eta_r + y)}]}{2|\beta|}$$

Now as  $\beta \rightarrow 0$  the  $\int_0^{\infty}$  is non singular and the

answer  $\rightarrow \frac{1}{2} (|\eta_r + y| - |\eta_r - y|)$  which is finite.

Hence:

$$\underline{S_T^{-1} \left( \frac{\sqrt{\frac{2}{\pi}} \sin(\alpha \eta_r)}{\alpha^2 + \beta^2} \right) = \frac{1}{2|\beta|} \left[ e^{-|\beta|(\eta_r - y)} - e^{-|\beta|(\eta_r + y)} \right]}$$

which holds for all  $\beta$ .

### (15.1.5) Inverse Complex Fourier Transform

The inverse complex Fourier transform is defined as:

$$F_T^{-1}(f(\beta)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\beta) e^{j\beta x} d\beta$$

$$(a) f(\beta) = \frac{1}{|\beta|} \left( e^{-|\beta|(\eta_r - y)} - e^{-|\beta|(\eta_r + y)} \right) e^{-j\beta \xi_r}$$

$$F_T^{-1}(f(\beta)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{|\beta|} \left( e^{-|\beta|(\eta_r - y)} - e^{-|\beta|(\eta_r + y)} \right) e^{j\beta(x - \xi_r)} d\beta$$

$$\text{Consider the integral } I_1 = \int_0^{\lambda_1} e^{-\lambda x} d\lambda = -\frac{1}{x} \left[ e^{-\lambda x} \right]_0^{\lambda_1} = \frac{1}{x} \left( 1 - e^{-\lambda_1 x} \right)$$

$$\text{Now consider } \int_{\lambda_2}^{\lambda_1} I_1 d\lambda = \log_e \left( \frac{\lambda_1}{\lambda_2} \right) = \int_{\lambda_2}^{\lambda_1} \int_0^{\infty} e^{-\lambda x} dx d\lambda$$

Reversing the order of integration gives:

$$\int_0^{\infty} \int_{\lambda_2}^{\lambda_1} e^{-\lambda x} d\lambda dx = \int_0^{\infty} \frac{e^{-\lambda_2 x} - e^{-\lambda_1 x}}{x} dx = \log_e \left( \frac{\lambda_1}{\lambda_2} \right)$$

Now:

$$F_T^{-1}(f(\beta)) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-|\beta(\eta_r-y)|} - e^{-|\beta(\eta_r+y)|}}{|\beta|} (\cos \beta(x-\xi_r) + j \sin \beta(x-\xi_r)) d\beta$$

Since the integrand of the imaginary part is odd:

$$\begin{aligned} &= \frac{1}{\pi} \operatorname{Re} \int_0^{\infty} \frac{e^{-\beta|\eta_r-y|} - e^{-\beta|\eta_r+y|}}{\beta} e^{j\beta(x-\xi_r)} d\beta \\ &= \frac{1}{\pi} \log_e \left| \frac{\lambda_1}{\lambda_2} \right| \end{aligned}$$

$$\begin{aligned} \text{where } \lambda_1 &= -|\eta_r+y| + j(x-\xi_r) \\ \lambda_2 &= -|\eta_r-y| + j(x-\xi_r) \end{aligned}$$

Therefore:

$$\begin{aligned} &\underline{F_T^{-1} \left( \frac{1}{|\beta|} (e^{-|\beta(\eta_r-y)|} - e^{-|\beta(\eta_r+y)|}) e^{-j\beta\xi_r} \right)} \\ &= \underline{\frac{1}{2\pi} \log_e \left[ \frac{(\eta_r+y)^2 + (x-\xi_r)^2}{(\eta_r-y)^2 + (x-\xi_r)^2} \right]} \end{aligned}$$

### (15.1.6) Inverse Complex Fourier Transform

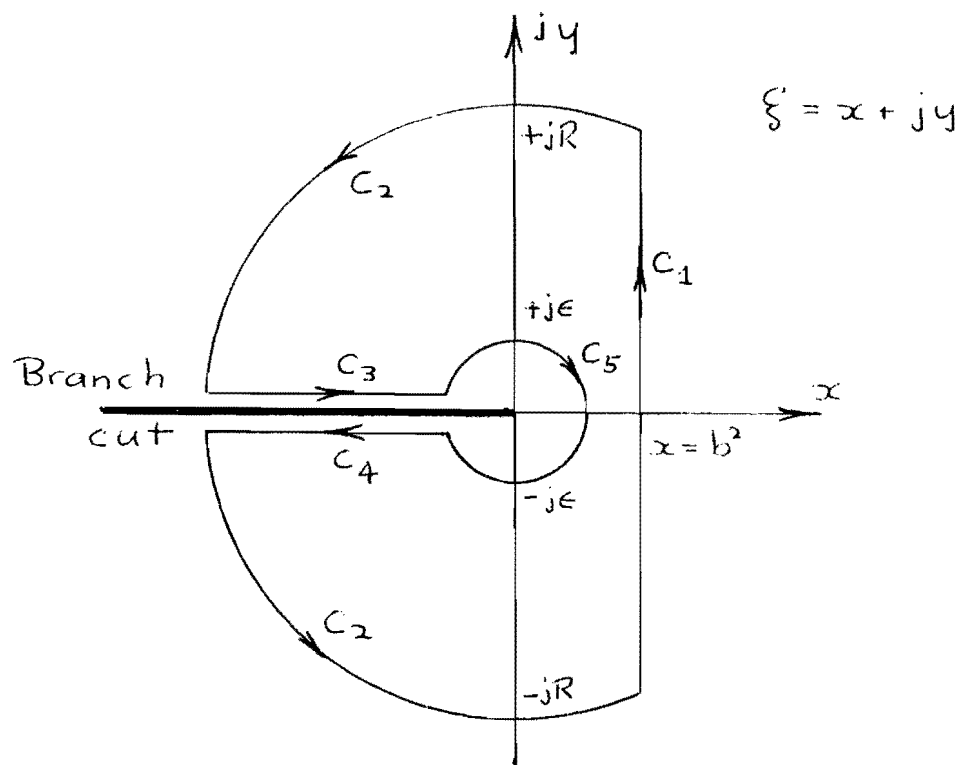
$$F_T^{-1} \left( \frac{1}{a + \sqrt{b^2 + j\omega c}} \right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{j\omega t}}{a + \sqrt{b^2 + j\omega c}} d\omega$$

Consider the case for  $a > 0$  and  $b > 0$ . Letting  $\xi = b^2 + j\omega c$  and substituting for  $\omega$  gives the integral:

$$\int_{b^2 - j\infty}^{b^2 + j\infty} \frac{e^{\frac{t(\xi - b^2)}{c}}}{a + \sqrt{\xi}} \cdot \frac{d\xi}{jc}$$

Letting  $\xi = re^{j\theta}$ ,  $\sqrt{\xi}$  has the values  $Z_1 = +\sqrt{r}e^{j\theta/2}$   
 $Z_2 = -\sqrt{r}e^{j\theta/2}$

From equation (15.6 a) section 15.4.2(c) and APPENDIX (15.2) it is seen that we require the positive square root. (i.e. Restrict the contour integration to the  $Z_1$  portion of the complex  $\xi$  plane.) We also require  $a + \sqrt{\xi}$  to be continuous along the line  $b^2 - j\infty$  to  $b^2 + j\infty$ . To do this place a branch cut along the negative real axis and restrict  $\theta$  to the values  $-\pi \leq \theta \leq \pi$ .



Integrate along the contour  $C = C_1 + C_2 + C_3 + C_4 + C_5$  in the complex  $\xi$  plane letting  $R \rightarrow \infty$  and  $\epsilon \rightarrow 0$ .

Along  $C_2$ , letting  $\xi = Re^{j\theta}$  gives:

$$\int_{\text{Along } C_2} \frac{e^{-\frac{b^2 t}{c}} e^{\frac{t}{c}(R \cos \theta + jR \sin \theta)}}{a + \sqrt{R^2} e^{j\theta/2}} \cdot \frac{Re^{j\theta}}{c} d\theta$$

As  $R$  tends large the  $e^{\frac{t}{c}R \cos \theta}$  term dominates  
 Provided that  $t > 0$ ,  $c > 0$ ,  $-\pi < \theta < -\frac{\pi}{2}$  and  $\frac{\pi}{2} < \theta < \pi$   
 the integrand  $\rightarrow 0$  as  $R \rightarrow \infty$  (for both  $z_1$  and  $z_2$ )

Along  $C_5$ , letting  $\xi = \epsilon e^{j\phi}$

$$\int_{\text{Along } C_5} \frac{e^{-\frac{b^2 t}{c}} e^{\frac{t}{c}(\epsilon \cos \phi + j\epsilon \sin \phi - b^2)}}{a^2 + \sqrt{\epsilon^2} e^{j\phi/2}} \cdot \frac{\epsilon e^{j\phi}}{c} d\phi$$

As  $\epsilon \rightarrow 0$ ,  $\int_{\text{Along } C_5} \rightarrow 0$ .

Along  $C_3$ ,  $\xi = -x = xe^{j\pi}$

$$\int_{\text{Along } C_3} = - \int_0^\infty \frac{e^{-\frac{t(x+b^2)}{c}}}{a + j\sqrt{x}} \cdot \frac{dx}{jc}$$

Along  $C_4$ ,  $\xi = -x = xe^{-j\pi}$

$$\int_{\text{Along } C_4} = \int_0^\infty \frac{e^{-\frac{t(x+b^2)}{c}}}{a - j\sqrt{x}} \cdot \frac{dx}{jc}$$

From which

$$\int_{C_3 \cup C_4} = \frac{2}{c} \int_{-\infty}^0 \frac{e^{-\frac{t(x+b^2)}{c}}}{a^2+x} \sqrt{x} dx$$

Putting  $x = y^2$  gives.

$$\int_{C_3 \cup C_4} = -\frac{4}{c} \int_0^{\infty} \frac{e^{-\frac{t(y^2+b^2)}{c}}}{a^2+y^2} y^2 dy$$

A pole can only occur when  $a = z_2$ . As the contour is restricted to the  $z_1$  branch of the complex  $s$  plane it encloses no poles. Hence:

$$\int_{-\infty}^{\infty} \frac{e^{j\omega t}}{a + \sqrt{b^2 + \omega^2 c}} d\omega = \frac{4}{c} e^{-\frac{tb^2}{c}} \int_0^{\infty} \frac{y^2}{y^2+a^2} e^{-\frac{ty^2}{c}} dy$$

$$\text{Now } \int_0^{\infty} \frac{y^2}{y^2+a^2} e^{-\frac{ty^2}{c}} dy = \int_0^{\infty} e^{-\frac{ty^2}{c}} dy - a^2 \int_0^{\infty} \frac{e^{-\frac{ty^2}{c}}}{y^2+a^2} dy$$

$$\text{and } \int_0^{\infty} e^{-\frac{ty^2}{c}} dy = \frac{1}{2} \sqrt{\frac{\pi c}{t}}$$

From ABRAMOWITZ (1968). (APPENDIX (13.2))

$$a^2 \int_0^{\infty} \frac{e^{-\frac{t}{c} y^2}}{a^2 + y^2} dy = \frac{\pi a}{2} e^{\frac{ta^2}{c}} \operatorname{erfc}\left(a \sqrt{\frac{t}{c}}\right)$$

So that

$$\underline{F_T^{-1}\left(\frac{1}{a + \sqrt{b^2 + \omega^2 c}}\right) = \frac{e^{-\frac{tb^2}{c}}}{\sqrt{\pi c t}} - \frac{a}{c} e^{\frac{ta^2}{c}} \operatorname{erfc}\left(a \sqrt{\frac{t}{c}}\right)}$$

which is valid for  $t > 0$ ,  $a > 0$ ,  $b > 0$  and  $c > 0$

Case for  $a=b=0$ . This corresponds to  $\beta=0$  and the integral to be evaluated is:

$$\int_{-\infty}^{+\infty} \frac{e^{j\omega t}}{\sqrt{j\omega c}} d\omega$$

Integrating this integral around the same contour as  $b \rightarrow 0$  it is found to tend to the limit of the previous result as  $\beta \rightarrow 0$ . Therefore the result holds for all  $\beta \geq 0$ .

Substituting  $a=\mu\beta$ ,  $b=\beta$ ,  $c=\mu\mu_0\sigma$  gives:

$$\begin{aligned} F_T^{-1} \left( \frac{1}{\mu\beta + \sqrt{\beta^2 + j\omega\mu\mu_0\sigma}} \right) \\ = \frac{e^{-\frac{t\beta^2}{\mu\mu_0\sigma}}}{\sqrt{\pi\mu\mu_0\sigma t}} - \frac{\beta}{\mu_0\sigma} e^{\frac{t\beta^2(\mu^2-1)}{\mu\mu_0\sigma}} \operatorname{erfc} \left( \beta \sqrt{\frac{t\mu}{\mu_0\sigma}} \right) \end{aligned}$$


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which is valid for  $t > 0$ ,  $\sigma > 0$ ,  $\beta > 0$



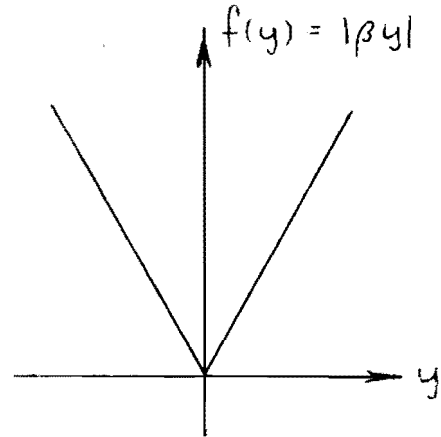
## APPENDIX (15.2)

DERIVATIVES OF FUNCTIONS FOR MULTIPLE  
TRANSFORM ANALYSIS

(a)  $\frac{dg}{dy}$  where  $g = e^{-|\beta y|}$

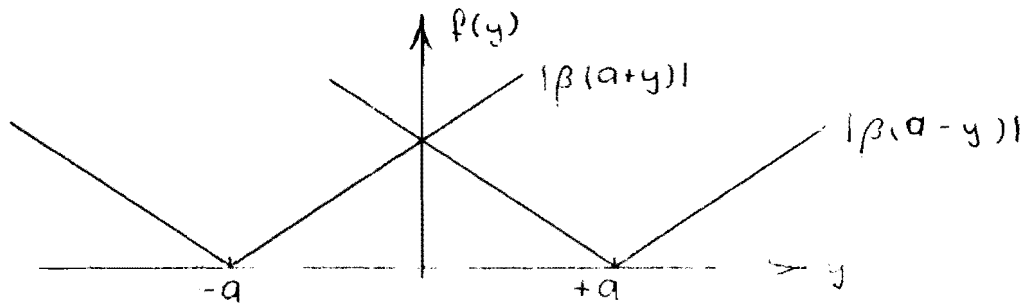
Let  $f(y) = |\beta y| = |\beta| |y|$

$$\frac{df}{dy} = \begin{cases} |\beta| & \text{for } y > 0 \\ -|\beta| & \text{for } y < 0 \end{cases}$$



giving  $\frac{dg}{dy} = \begin{cases} -|\beta| e^{-|\beta y|} & y > 0 \\ +|\beta| e^{-|\beta y|} & y < 0 \end{cases}$

(b)  $\frac{dg}{dy}$  where  $g = e^{-|\beta(a-y)|}$  and  $g = e^{-|\beta(a+y)|}$



Let  $f(y) = |\beta(a-y)|$

$$\frac{df}{dy} = \begin{cases} |\beta| & y > a \\ -|\beta| & y < a \end{cases}$$

giving  $\frac{dg}{dy} = \begin{cases} -|\beta| e^{-|\beta(a-y)|} & y > a \\ +|\beta| e^{-|\beta(a-y)|} & y < a \end{cases}$

Similarly for  $g = e^{-|\beta(a+y)|}$

$$\frac{dg}{dy} = \begin{cases} -|\beta| e^{-|\beta(a+y)|} & \text{for } \begin{cases} y > -a \\ y < -a \end{cases} \end{cases}$$

## APPENDIX (15.3)

INTEGRATION OVER  $\beta$ 

It is required to carry out the integration:

$$\text{Int} = \int_0^{\infty} g(\beta, T) e^{-2h\beta} d\beta$$

where

$$g(\beta, T) = \frac{e^{-\frac{T\beta^2}{\mu_g \sigma}}}{\sqrt{\pi \mu_g \sigma T}} - \frac{\beta}{\mu_0 \sigma} e^{-\frac{T(\mu_r^2 - 1)\beta^2}{\mu_g \sigma}} \cdot \text{erfc}\left(\beta \sqrt{\frac{T\mu_r}{\mu_0 \sigma}}\right)$$

Considering the first term of  $g(\beta, T)$ , the integration required is:

$$\text{Int}_1 = \int_0^{\infty} e^{-(a\beta^2 + b\beta)} d\beta$$

From ABRAMOWITZ (1968, integral 7.4.2):

$$\text{Int}_1 = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}} \text{erfc}\left(\frac{b}{2\sqrt{a}}\right), \quad a \neq 0$$

which on substituting for 'a' and 'b' yields

$$\int_0^{\infty} \frac{e^{-\left(\frac{T\beta^2}{\mu_g \sigma} + 2h\beta\right)}}{\sqrt{\pi \mu_g \sigma T}} d\beta = \frac{e^{-\frac{h^2 \mu_g \sigma}{T}} \cdot \text{erfc}\left(h \sqrt{\frac{\mu_g \sigma}{T}}\right)}{2T}, \quad T > 0$$

The second integration required is:

$$\text{Int}_2 = \int_0^{\infty} \beta e^{-\frac{T(\mu_r^2 - 1)\beta^2}{\mu_0 \sigma}} \text{erfc}\left(\beta \sqrt{\frac{T\mu_r}{\mu_0 \sigma}}\right) e^{-2h\beta} d\beta$$

Evaluation of this integral is considerably simplified if the relative permeability of the ground is set equal to unity, i.e.  $\mu_r = 1$ . As pointed out in section 15.2.3) this is usually of no consequence and is a frequently made simplifying assumption. Thus we are required to evaluate:

$$\begin{aligned} \text{Int}_2 &= \int_0^{\infty} \beta e^{-a\beta} \text{erfc}(b\beta) d\beta \\ &= -\frac{d}{da} \int_0^{\infty} e^{-a\beta} \text{erfc}(b\beta) d\beta \end{aligned}$$

In its last form the integral is found from ABRAMOWITZ (1968, integral 7.4.17) giving:

$$\begin{aligned} \text{Int}_2 &= -\frac{d}{da} \left\{ \frac{1}{a} \left\{ 1 - e^{\frac{a^2}{4b^2}} \text{erfc}\left(\frac{a}{2b}\right) \right\} \right\} \\ &= \left\{ \frac{1}{2b^2} - \frac{1}{a^2} \right\} e^{\left(\frac{a}{2b}\right)^2} \text{erfc}\left(\frac{a}{2b}\right) + \frac{1}{a^2} - \frac{1}{ab\sqrt{\pi}} \\ &\quad \text{for } a > 0, b > 0 \end{aligned}$$

Substituting for 'a' and 'b' gives:

$$\begin{aligned} &\frac{1}{\mu_0 \sigma} \int_0^{\infty} \beta e^{-2h\beta} \text{erfc}\left(\beta \sqrt{\frac{T}{\mu_0 \sigma}}\right) d\beta \\ &= \left\{ \frac{1}{2T} - \frac{1}{4\mu_0 \sigma h^2} \right\} e^{h^2 \frac{\mu_0 \sigma}{T}} \text{erfc}\left(h \sqrt{\frac{\mu_0 \sigma}{T}}\right) - \frac{1}{4\mu_0 \sigma h^2} \\ &\quad + \frac{1}{2h\sqrt{\pi\mu_0 \sigma T}} \end{aligned}$$

Substituting for  $\text{Int}_1$  and  $\text{Int}_2$  gives

$$\text{Int} = \left\{ \frac{2h\sqrt{\mu_0\sigma}}{\sqrt{\pi T}} + e^{\frac{h^2\mu_0\sigma}{T}} \cdot \text{erfc} \left( h\sqrt{\frac{\mu_0\sigma}{T}} \right) - 1 \right\} / (4\mu_0\sigma h^2)$$

which is valid for  $T > 0$ .

## SECTION 16

SOLUTION OF TRAVELLING WAVE PROBLEMS ON TRANSMISSION  
SYSTEMS WITH FREQUENCY DEPENDENT PARAMETERS

In section 15) the effect on the transmission line parameters of finite conductivity in the line conductors was formulated. In particular the situation where the earth forms part of the circuit was considered. It was found that finite conductivity gave rise to transmission line parameters that were a function of frequency. For the power system situation the parameter most affected was the series impedance where the finite conductivity of the earth gave rise to both a circuit resistance and inductance that were functions of frequency. Not surprisingly, in view of the above, the most common formulation of these effects is in terms of a steady state sinusoidal travelling wave. The solution of problems involving steady state sinusoidal signals on earth return transmission systems is thus readily achieved using a constant parameter transmission line representation exactly as in section 4) where the line parameters are calculated at the signal frequency using one of the formulations given in section 15).

The method of modal components can still be used for multiple circuit transmission lines. The effect of the earth's finite conductivity on component transmission systems is, however, not constant. The line modes of propagation (where the signal return path is in the other wires comprising the system) are not greatly affected by the presence of the earth. The ground mode (where the signal

return path is in the earth) is greatly affected by the earth's finite conductivity modifying its velocity of propagation, attenuation and distortion. Furthermore, the earth's effects depend mainly on the height of the conductor and on the earth resistivity but the conductor size may be neglected (SUNDE, 1968). Thus when a surge is transmitted over a group of parallel conductors with an earth return it will be distorted and attenuated to practically the same extent as when transmitted over a single wire at the same height. Herein lies the rationale for the great emphasis on the transmission system consisting of a single overhead conductor with a finitely conducting earth return. The effect of the earth on a multiple circuit transmission line can be closely approximated by transforming into an uncoupled modal component system, treating the line modes as being unaffected by the earth and solving the ground mode as for a single overhead conductor with an earth return. This has become the dominant approach to the solution of transmission problems on such lines.

As with the rest of this analysis, transient problems are of primary concern. Furthermore, as already pointed out, the solution of steady state sinusoidal problems present no difficulties. The solutions considered here are thus concerned with travelling wave surge problems. For reference they have been classified into four types.

(1) Frequency domain solutions	{ BATTISON 1969 * WEDEPOHL 1969 * HYLTEN-CAVALLIUS 1959 BROWN 1971
(2) Operational solutions	{ SHAH 1971 MAGNUSSON 1968 SUNDE 1968 ROBERT, TRAN, TRAN 1969, 1972 UMOTO 1971 HOLT 1970
(3) Lumped circuit element approximations	{ DOLGINOV 1966 * STUPEL 1972 *
(4) Time domain techniques	{ McELROY 1963 WAGNER 1963 BICKFORD 1967 * BUNDER 1970 * SNELSON 1972 *

In these classifications the lumped circuit element approximations have been classified separately from time domain solutions. However, although they can be used for both frequency domain and time domain solutions their expressly stated purpose is for use in the time domain.

The most frequently used formulation in these solutions is CARSON'S (1926) formulation. However, a number of the solution methods allow any particular frequency dependence to be specified at the time of application to a problem. These have been marked with an asterisk \*.

## 16.1) FREQUENCY DOMAIN SOLUTIONS

By its very nature the problem of transmission with frequency dependent parameters is particularly suited to solution using frequency domain techniques. The most complete solutions for frequency dependent transmission systems have been produced by BATTISSON (1969) and WEDEPOHL (1969). In these, multiple circuit transmission lines are handled by a further transform in the case of BATTISSON and by modal components in the case of WEDEPOHL. In both cases the frequency dependence of all modes of propagation is included (not just the earth mode). Both use the modified Fourier transform technique outlined in section 4.3.1). HYLTEN-CAVALLIUS (1959) and BROWN (1971) are both concerned with finding the step response of an overhead conductor with a ground return although HYLTEN-CAVALLIUS later uses this to modify the earth mode of a multiple circuit transmission line.

### 16.1.1) BATTISSON (1969)

BATTISSON states (and correctly so relative to frequency domain methods) that use of the Fourier transform overcomes the difficulties associated with frequency dependent parameters without any increase in computation times. Transforming w.r.t. time reduces the problem to a set of ordinary differential equations a solution of which can be effected by either transform methods or by separating into modal components. BATTISSON uses a further transformation to a set of algebraic equations. The transform used to remove time as a variable is the 'Modified Fourier Transform'



(discussed in section 4.3.1)) and the inverse transform of the solution is computed numerically.

This paper examines some of the effects of the frequency dependence of transmission line parameters. The earth effect formulation used is that of WEDEPOHL (1966). Illustrative examples are considered of single phase energization of multiphase lines. In the course of these examples it is shown that:

- (i) Use of geometric mean distances for bundled conductors above a finitely conducting earth introduces negligible error.
- (ii) For horizontally stratified earth structures the line inductance depends mostly on the lower strata resistivities at low frequencies ( $< 10^2$ - $10^3$  Hz) and the upper layer resistivity at higher frequencies ( $> 10^5$  Hz).
- (iii) Although high frequency inductance is sensitive to earth stratification the initial rate of rise of a surge seems to be little affected. The responses seem sensitive to earth stratification only at those times when intermediate frequencies become effective. The effect of earth stratification on the low frequency components of a surge becomes apparent only after a long time (3.5 milli-seconds in the example considered).
- (iv) The effect of frequency dependence on conditions under which resonance of long lines can be expected gives at line lengths of less than a quarter wavelength higher

overvoltages than are predicted by the corresponding lossless line.

- (v) Fixed parameter representation is inadequate for solving surge problems on transmission lines with frequency dependent parameters, even when the parameters are computed at the dominant frequency.

#### 16.1.2) WEDEPOHL (1969)

Also uses the modified fourier transform technique but separates the resulting ordinary differential equations into an uncoupled set of modal equations which are then solved as for a single transmission line (Fig. (4.2)). This procedure is adopted for each frequency component. As in BATTISSON (1969) single and three phase energization of a three phase transmission line are considered. Recovery voltage calculations are also included. Earth effects are included with CARSON'S (1926) formulation and a uniform earth with a resistivity of 30 ohm metres is used.

#### Comment on BATTISSON and WEDEPOHL (1969)

At this point in time fourier transform methods had not been developed to the point where non-simultaneous closure or opening of circuit breakers could be considered. Subsequent developments (BATTISSON (1970) and WEDEPOHL (1970)) eventually made this possible.

### 16.1.3) HYLTEN-CAVALLIUS (1959)

Here the fourier transform of the step response of a single overhead line with an earth return is formulated. The inverse numerical transform from the frequency domain to the time domain is accomplished by a graphical piecewise smooth approximation that is claimed to be highly accurate. The effects of the ground are included by CARSON'S (1926) formulation and the line response is formulated both with and without aerial earth return wires. The fourier transformed step response without aerial earth wires is formulated as:

$$A(\omega) = \frac{e^{j\omega T_1}}{j\omega} e^{-j\omega T_1} \left\{ 1 + \frac{120}{jZ_B} [P(r) + jQ(r)] \right\}^{\frac{1}{2}}$$

where  $[P(r) + jQ(r)]$  are CARSON'S correction terms

$Z_B = \log \frac{2h}{a}$  is the wave impedance

$T_1$  and  $\omega$  are respectively normalized travel time and frequency. Normalized w.r.t. the time factor

$$\beta^2 = h^2 \mu_0 / \rho \text{ micro seconds.}$$

This is calculated for each frequency and the inverse fourier transform found from:

$$F(t) = \frac{2}{\pi} \int_0^\infty -\mathcal{J}_m[A(\omega)] \sin(\omega t) d\omega$$

by a graphical calculation in which the curve  $f = -\mathcal{J}_m[A(\omega)]$  is drawn and approximated by a sum of simple analytical functions

$$\begin{array}{ll}
 f_1(\omega) & a < \omega < b \\
 f_2(\omega) & c < \omega < d \\
 \vdots & \\
 f_n(\omega) & i < \omega < j
 \end{array}$$

so that the whole interval  $0 < \omega < \infty$  is covered.

Multiple circuit transmission lines are treated by modifying the earth component of propagation. Also covered in this paper are field tests, distortion due to corona and comparisons between measurement and calculation.

#### 16.1.4) BROWN (1971)

Uses the fourier integral representation of a unit step function combined with the d.c. and steady state sinusoidal response of a transmission line to obtain the step response of a single overhead line with an earth return. He uses his own formulation (an extension of CARSON'S formulation to include the relative permeability of the earth) to include the effects of the earth's finite conductivity (section 15.2.1)(c)). The solution is applied to a three phase fully transposed transmission line by solving for the zero sequence component which is formulated both with and without aerial earth return wires. The required integrals are computed numerically.

The unit step function is represented by:

$$U(t) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{\sin(\omega t)}{\omega} d\omega$$

Substituting for the d.c. and sinusoidal responses of the transmission line gives the step response:

$$V(z,t) = \frac{1}{2} e^{-\sqrt{RG}z} + \frac{1}{\pi} \int_0^{\infty} \frac{e^{-\alpha z} \sin(\omega t - \frac{z}{v})}{\omega} d\omega$$

where  $v = \frac{\omega}{\beta}$  is the phase velocity.

For the case where  $G = 0$  the response is finally given by:

$$V(z,t) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{e^{-\alpha z} \sin(\omega T - \theta)}{\omega T} d(\omega T)$$

where  $\theta = \beta z - \omega T_1$ ,  $T = t - T_1$ ,  $\alpha = \text{Re}(\gamma)$ ,  $\beta = \mathcal{I}_m(\gamma)$

$\gamma = [j\omega C(Z + \Delta Z)]^{\frac{1}{2}}$  where  $C$  and  $Z$  are respectively the capacitance and inductance with a perfectly conducting ground.

$\Delta Z = \frac{\omega\mu}{\pi} [P + jQ]$  takes into account the earth's finite conductivity with:

$$P = \int_0^{\infty} \frac{e^{-rw} \sin(rw)}{\sqrt{w^2 + 1} + w} dw, \quad Q = \int_0^{\infty} \frac{e^{-rw} \cos(rw)}{\sqrt{w^2 + 1} + w} dw$$

For the zero sequence component of a fully transposed three phase line the propagation constant becomes:

$$\gamma = j\omega\sqrt{\mu\epsilon} \left[ 1 + \frac{2}{j} \frac{(P + jQ)}{\log \frac{GMP}{GMR}} \right]^{\frac{1}{2}}$$

The required integrals are evaluated numerically using Romberg integration.

## 16.2) OPERATIONAL SOLUTIONS

These solutions are concerned with obtaining time domain expressions for the step or impulse response of a single circuit line with an earth return. Usually it is attempted to compose these expressions of simple mathematical functions. Often to facilitate simple inversion from the operational expressions to the time expressions approximations are made which give results valid only for small or large values of times. The formulas of SHAH (1971), MAGNUSSON (1968) and WIGINTON (1957) have all been derived from expressions for large  $s$  (or 'p') and are therefore valid only for short values of time.

### 16.2.1) SHAH (1971)

SHAH sets out to solve the circuit shown in Fig. (16.1). He uses his own formulation (section 15.3.3) (a)) for the series impedance of the transmission line.

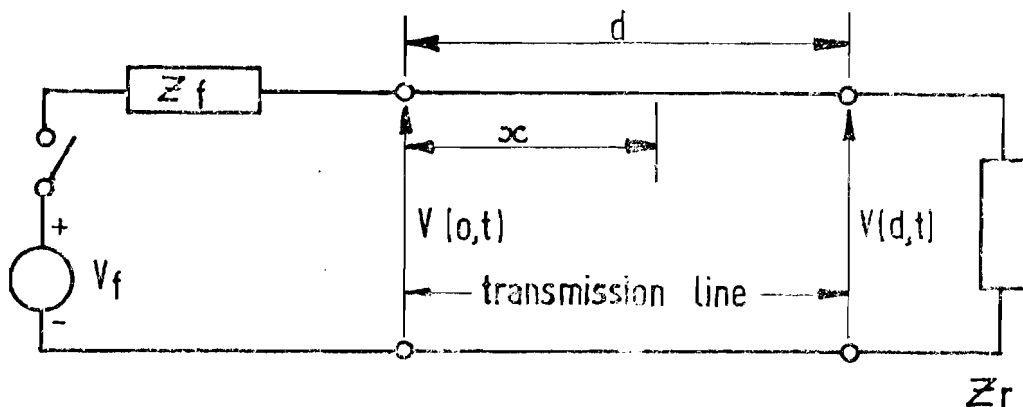


Fig. (16.1) Circuit analysed by SHAH (1971)

The operational equation used is:

$$v(d,s) = \frac{z_0(s)}{z_0(s)+z_f(s)} \cdot V_f(s) \cdot \frac{e^{-\gamma(s)d} + \lambda_r(s)e^{-\gamma(s)d}}{1 - \lambda_r(s)\lambda_g(s)e^{-2\gamma(s)d}}$$

where  $\lambda_r, \lambda_g = \frac{z_{r,g}(s) - z_0(s)}{z_{r,g}(s) + z_0(s)}$  are reflection coefficients,

$$\gamma(s) = s\sqrt{LC} + K_4\sqrt{s}, \quad z_0(s) = \sqrt{\frac{1}{C}(L + \frac{\alpha+\beta}{\sqrt{s}})} \quad \text{and}$$

$$K_4 = \frac{\alpha+\beta}{2} \sqrt{\frac{C}{L}}. \quad (\gamma(s) \text{ is derived from } \sqrt{sC(sL+R+(\alpha+\beta)\sqrt{s})}$$

by neglecting  $R$ , expanding binomially and selecting the first two terms.)

By neglecting reflection coefficients, setting  $z_f(s)$  to zero and taking inverse Laplace transforms one obtains:

$$v(d,t) = V_{dc} \operatorname{erfc} \left( \frac{K_4 d}{2\sqrt{t - \frac{d}{c}}} \right)$$

for  $t \geq \frac{d}{c}$  where  $c$  is the velocity of light.

The physical and mathematical machinations that SHAH resorts to in achieving this straightforward result (which was derived at least 14 years previously: WIGINGTON (1957)) are quite remarkable.

Note: For the circuit given the first reflected wave at least cannot be neglected as it is produced coincident with the incident wave. The case for an infinite line and  $z_f(s) = R_g \neq 0$  has also been solved previously (MAGNUSSON, 1968).

16.2.2) MAGNUSSON (1968)

MAGNUSSON solves for the case of a step of voltage applied through a source resistor,  $R_g$ , to an infinite line with finite conductivity. The operational equations are ( $V_{dc} = V_f$  in Fig. (16.1)):

$$I(x,s) = I(0,s) e^{-\gamma(s)x}$$

$$I(0,s) = \frac{V_{dc}}{s} \cdot \frac{1}{(R_g + Z_0(s))}$$

$$V(0,s) = \frac{V_{dc}}{s} - R_g \cdot I(0,s)$$

$$V(x,s) = V(0,s) e^{-\gamma(s)x}$$

$$\text{where } \gamma(s) = s \cdot \sqrt{LC} + \frac{K}{2} \sqrt{\frac{L}{C}} \cdot \sqrt{s}, \quad Z_0(s) = \sqrt{\frac{K\sqrt{s} + sL}{sC}}$$

$$\text{and } K = \frac{1}{2\pi r} \sqrt{\frac{\mu}{\sigma}}.$$

Taking inverse transforms gives:

$$v(x,t) = V_{dc} \cdot \text{erfc}(T_x / 2\sqrt{t-T_x}) \cdot U(t-T_x) - R_g \cdot i(x,t)$$

$$i(x,t) = \left\{ \frac{V_{dc}}{R_g + \sqrt{\frac{L}{C}}} \right\} e^{(K^2(t-T_x) + Kt)} \cdot \text{erfc} \left( K\sqrt{t-T_x} + \frac{T_x}{2\sqrt{t-T_x}} \right) \cdot U(t-T_x)$$

$$\text{where } K = \sqrt{\frac{L}{C}} / (R_g + \sqrt{\frac{L}{C}}) \quad \text{and} \quad T_x = \frac{x}{c}.$$



16.2.3) SUNDE (1968)

SUNDE derives expressions for a single overhead conductor with an earth return for both small and large values of 'p' (section 15.3.3)(c)). For large values of 'p' he obtains the operational expression:

$$\frac{I(x,p)}{I(o,p)} = e^{-p\frac{x}{c}} e^{\sqrt{p} x (2c\beta_1 \log \frac{2h}{a})^{-1}}$$

which for a step input inverts to give:

$$i(x,t) = \text{erfc} \left[ x (4\beta_1 c \sqrt{t_0} \log \frac{2h}{a})^{-1} \right]$$

and which is valid for small  $t_0$  where  $t_0 = t - \frac{x}{c} \geq 0$ .

For small values of 'p' SUNDE obtains:

$$\frac{I(x,p)}{I(o,p)} = e^{-p\frac{x}{c}} \left[ 1 + \frac{px}{4c \log \frac{2h}{a}} (\log \beta^2 p - \frac{2}{\beta_i \sqrt{p}}) \right]$$

which for a step inverts to:

$$i(x,t) = 1 - x \frac{1 + (\frac{2}{\beta_i}) \sqrt{\frac{t_0}{\pi}}}{t_0 4c \log \frac{2h}{a}}$$

and which is valid for large values of  $t_0$ .

Typically small values of  $t_0$  would be up to half a micro-second and large values of time would be greater than 3-4 micro-seconds.

Comment: From the above it is evident that formulas for small values of time do not adequately represent the entire wavefront and should not be used as the sole correction for studies of travelling waves with earth returns.

#### 16.2.4) ROBERT (1969), TRAN (1972), TRAN (1972)

ROBERT and TRAN use their own formulation of finite earth and conductor conductivity effects (section 15.3.3) (b)). They develop a single expression which is valid over the entire time span of interest in many surge propagation problems. Their earth formulation is extended to multiple circuit transmission lines and in a form valid for small and large 'p' is shown to be in reasonable agreement with CARSON'S (1926) formulation. These results are then applied to the solution of surges on mutually coupled multiple circuit transmission lines. In a comparison of actual oscillograms and calculated results they show a close correspondence between the two.

The propagation constant used is  $\gamma = p(1 + \frac{a_{cl} + a_g}{2p})$  which when developed for large values of 'p' gave  $\gamma = p + \beta_0\sqrt{p} + \alpha_0$ . This substituted into  $(V_0 C/p)e^{-\gamma \frac{x}{c}}$  yielded on inversion:

$$q_0(t_0) = CV_0 e^{\alpha_0 \frac{x}{c}} \operatorname{erfc} \left( \frac{\beta_0 x}{2c\sqrt{t_0}} \right)$$

which is valid for small values of  $t_0 = t - \frac{x}{c} \geq 0$ .

For intermediate values of 'p' the conductor correction is developed for large values of 'p' and the ground correction for small values of 'p'. This eventually yields  $\gamma = p(1+T_m) + \beta_f\sqrt{p} + \alpha_f$  which on inversion gives the formula valid for intermediate times  $T_1 \leq t_0 \leq T_2$ :

$$q_f(t_0') = CV_0 e^{-\alpha_f \frac{x}{c}} \operatorname{erfc} \left( \frac{\beta_f x}{2c\sqrt{t_0'}} \right)$$

where  $t_0' = t_0 - T_m \cdot \frac{x}{c} > 0$ . Here  $T_m$  is a delay term introduced by approximating a log variable by a straight line over a range of its arguments. It will be seen that the formulas for short and intermediate times have the same mathematical form. By letting the range of arguments over which the straight line log approximation holds tend to zero, the two formulas can be combined giving:

$$q(t_0) = CV_0 e^{-\alpha_f \frac{x}{c}} \operatorname{erfc} \left( \frac{\beta_i x}{2c \sqrt{t_0 - T_i \frac{x}{c}}} \right)$$

$$\text{where } T_i = \frac{c}{4\pi\epsilon_0} \left[ \log \frac{1}{h} \sqrt{\frac{Kt_0}{\mu_0 \sigma_2}} - 1.0772 \right] = T_i(t_0) \rightarrow 0 \quad \text{as } t_0 \rightarrow 0$$

$$\beta_i = \frac{c}{4\pi\epsilon_0 \sqrt{\mu_0}} \left[ \frac{\sqrt{\rho_1}}{r_c} + \sqrt{\frac{\mu_0}{Kt_0}} \right]$$

and the constant  $K$  is determined from the theoretical curves.

#### 16.2.5) UMOTO (1971)

UMOTO finds the general time response of an overhead wire with an earth return. As the applied voltage is not specified this solution takes a convolution form. The operational equation is the now familiar:

$$V(x,p) = e^{-\frac{x}{c}p} e^{-Kx\sqrt{p}} E(p)$$

where  $K = (\alpha + \beta)/2Z$ ,  $\alpha = \sqrt{(\mu\rho/\pi) 10^{-9}/r}$  (conductor distortion factor),  $\beta = \sqrt{(2\mu_g\rho_g/\pi) 10^{-9}/h}$  (ground distortion factor) and  $Z = \sqrt{\frac{L}{C}}$ .

This inverts to:

$$v(x,t) = H\left(t - \frac{x}{c}\right) \int_{\frac{x}{c}}^t e(t-T) \left\{ \frac{d}{dT} \operatorname{erfc} \left( \frac{Kx}{2\sqrt{T-\frac{x}{c}}} \right) \right\} dT$$

Setting  $e(t) = E_0 e^{-at}$  gives:

$$v(x,t) = H\left(t - \frac{x}{c}\right) E_0 \frac{Kx}{2\sqrt{\pi}} \int_0^{t-\frac{x}{c}} F(T') dT'$$

where  $T' = T - \frac{x}{c}$ .

As stated, the above integral is not very suited to numerical integration because of the nature of  $F(T')$  and the wide range of integration. By changing the variable of integration to  $t''$  ( $T' = 10^{t''}$ ) and noting that  $\int_0^{10^{-11}} F(T') dT' \approx 0$ , UMOTO converts it into a form more suitable for numerical integration:

$$v(x,t) = H\left(t - \frac{x}{c}\right) E_0 \log_{10} 10 \frac{\sigma x}{2\sqrt{\pi}} \int_{-11}^{\log_{10} \left(t - \frac{x}{c}\right)} F(10^{t''}) 10^{t''} dt''$$

#### 16.2.6) HOLT (1970)

This paper is not concerned with propagation along power system transmission lines but with skin effect on lossy coaxial cables. However, the operational equations met in the earth return power system situation usually have the same mathematical form. (SHAH'S (1971) circuit equations are identical.) The analysis given here is more rigorous in its mathematics than is usually the case in power systems references and so has been included for that reason. The

circuit solved for is shown in Fig. (16.1). As well as the step response for an infinite line (cited here) the responses for the cable open circuited and for the impedance values  $Z_f(s) = Z_r(s) = \sqrt{\frac{L}{C}}$  are given. In comparisons with experimental results HOLT shows the analysis to be very accurate.

The skin effect is formulated as  $Z(s) = Ls + K\sqrt{s} + R$  with  $Y(s) = Cs$ . The propagation constant  $\gamma(s)$  is expanded binomially and rearranged giving:

$$\gamma(s) = s \left\{ 1 + \sum_{n=0}^{\infty} a_{1,n+1}^{(1)} s^{-\frac{n+1}{2}} \right\}$$

$$\text{where } a_{1,2n}^{(1)} = (-1)^{n-1} \sum_{m=0}^n (-1)^m c_{n+m-1} d_{n+m,2m}^{(1)}$$

$$a_{1,2n+1}^{(1)} = (-1)^n \sum_{m=0}^n (-1)^m c_{n+m} d_{n+m+1,2m+1}^{(1)}$$

$$c_n = \left| \frac{1}{2}(-\frac{1}{2}) \cdots (\frac{1}{2}-n) \right| \cdot \frac{1}{(n+1)!}$$

$$d_{n,k}^{(1)} = \frac{n!}{(n-k)!k!} \beta_0^{n-k} \beta_2^k$$

$$\beta_0 = \frac{R}{L} \ell \sqrt{LC}, \quad \beta_2 = \frac{K}{L} \sqrt{\ell} (LC)^{\frac{1}{4}} \quad \text{and} \quad s = s \ell \sqrt{LC}$$

( $\ell = d$  in Fig. (16.1))

which is valid for frequencies

$$\omega > |s| > S_2 = \left[ \frac{K}{2L} + \sqrt{\frac{K^2}{4L^2} + \frac{R}{L}} \right]^2.$$

For low frequencies  $0 < |s| \leq S_0 < \text{minimum} \left( \frac{R}{L}, \frac{R^2}{4K^2} \right)$

$\gamma(s)$  is expanded as:

$$\gamma(s) = \sqrt{RC} \sqrt{s} \left\{ 1 + \sum_{n=0}^{\infty} a_{1,n+1}^{(2)} s^{\frac{n+1}{2}} \right\}$$

$$\text{where } a_{1,2n}^{(2)} = (-1)^{n-1} \sum_{m=0}^n (-1)^m c_{n+m-1} d_{n+m,n-m}^{(2)}$$

$$a_{1,2n+1}^{(2)} = (-1)^n \sum_{m=0}^n (-1)^m c_{n+m} d_{n+m+1,n-m}^{(2)}$$

$$d_{n,k}^{(2)} = \frac{n!}{(n-k)!k!} \eta_0^{n-k} \eta_2^k, \quad \eta_0 = \frac{K}{R}, \quad \eta_2 = \frac{L}{R}$$

The function  $A(w)e^{\alpha(x)g(w)}$  is expanded in a Taylor expansion giving the series:

$$A(w)e^{\alpha(x)g(w)} = \sum_{n=0}^{\infty} P_n(\alpha) w^n$$

which is then inverted term by term. For the step response of an infinite line the results obtained are:

(i) For values  $1 \leq t \leq T_0$  with  $E = V_{dc} \ell \sqrt{LC}$

$$v(\ell, t) = E e^{-a_{1,2}^{(1)}} \sum_{n=0}^{\infty} b_{n,1}^{(2)} \{4(T-1)\}^{\frac{n}{2}} I^n \operatorname{erfc}\left\{\frac{a_{1,1}^{(1)}}{2}(t-1)^{-\frac{1}{2}}\right\}$$

where  $I^n \operatorname{erfc}(u = \frac{K}{2\sqrt{t}})$  is the  $n^{\text{th}}$  repeated integral of  $\operatorname{erfc}(x)$  and possesses the recursion relation:

$$I^n \operatorname{erfc}(u) = \int_u^{\infty} I^{n-1} \operatorname{erfc}(x) dx.$$

$T_0$  satisfies  $\{4(T_0-1)\}^{\frac{n}{2}} I^n \operatorname{erfc}\left\{\frac{a_{1,1}^{(1)}}{2}(T_0-1)^{-\frac{1}{2}}\right\} = S_4^{\frac{n}{2}}$  with  $S_4 > S_2 \ell \sqrt{LC}$ .

(ii) For large values of time:

$$v(l,t) = E \operatorname{erfc}(\delta t^{-\frac{1}{2}})$$

$$+ \frac{E}{\sqrt{\pi}} e^{-\delta^2 t} \left\{ \sum_{n=0}^{\infty} b_{2n+1,1}^{(2)} \cdot \frac{H_{2n}(\delta t^{-\frac{1}{2}})}{2^{2n} t^{\frac{2n+1}{2}}} + b_{2n+2,1}^{(2)} \cdot \frac{H_{2n+1}(\delta t^{-\frac{1}{2}})}{2^{2n+1} t^{\frac{n+1}{2}}} \right\}$$

where  $\delta = \frac{l}{2} \sqrt{RC}$  and  $H_n$  denotes a Hermite polynomial of degree  $n$ .

### 16.3) LUMPED CIRCUIT ELEMENT APPROXIMATIONS

This approach is a more refined version of the Discrete Lumped Loss Approximation method outlined in section 4.7.5). Here the transmission line is represented as shown in Fig. (16.2).

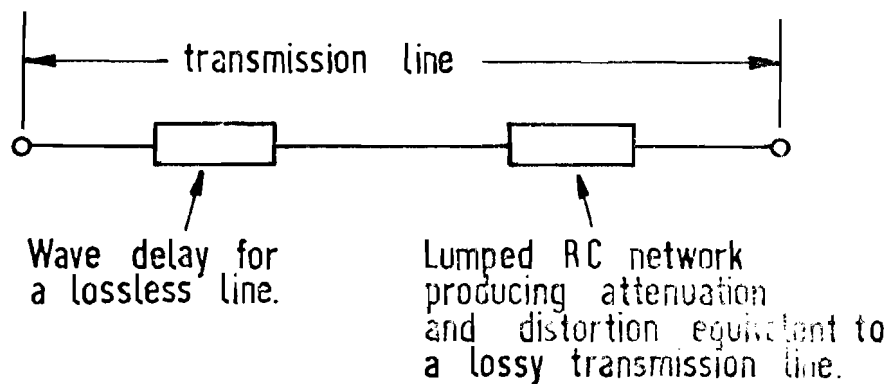


Fig. (16.2) Transmission Line Representation

The major attribute of this approximation is the ease with which it can be incorporated into time domain solutions of transient problems.

### 16.3.1) DOLGINOV (1966)

DOLGINOV considers transient surge problems in power systems. He simulates a fully symmetric three phase transmission line by separating into an uncoupled component system, attenuating the line components exponentially and solving the earth component as in Fig. (16.2). The R.C. network used is shown in Fig. (16.3) where the component values were chosen to match the frequency response of CARSON'S formulation.

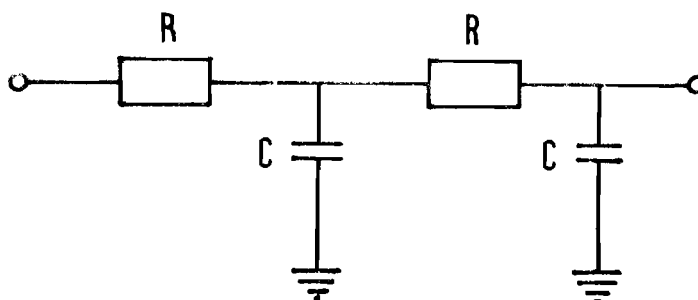


Fig. (16.3) R.C. network used by DOLGINOV to take the finite conductivity of the earth into account.

In his analysis DOLGINOV approximates lumped capacitive elements by open circuited stub transmission lines with characteristic impedance  $Z_c = \frac{\Delta T}{2X_c}$  and length  $l = \frac{\Delta T}{2}$  where  $\Delta T$  is the solution time increment. This approximation is discussed more fully in BICKFORD (1967) and is a method frequently used in time domain travelling wave solutions to overcome the need for integration. A similar approach is used to represent lumped inductances.



16.3.2) STUPEL (1972)

STUPEL refines the approximation used by DOLGINOV. The transfer function of the line is matched to an R.C. network by taking its logarithm and assuming it to be composed of straight lines. This effectively gives the asymptotic form of a Bode plot to which a network is matched by considering its break points and examining its phase. In circuit terms, the amplitude frequency characteristics of the line are approximated by the frequency characteristics of the units  $1+j\omega T$  and  $1/(1+j\omega T)$ .

For the earth mode of a two pole d.c. transmission line 500 km in length, the circuit obtained by matching to the response corrected according to CARSON'S formulation (as shown in Fig. (2) of STUPEL'S paper) is given in Fig. (16.4). This circuit closely follows the response corrected according to CARSON in both amplitude and phase up to about 2 KHz.

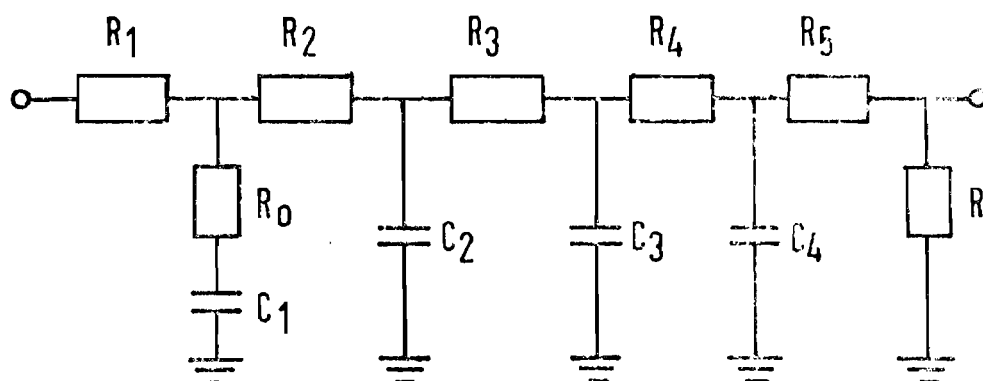


Fig. (16.4) R.C. network for earth mode of d.c. transmission line.

#### 16.4) TIME DOMAIN SOLUTIONS

Time domain solutions are inherently less suited to solving problems involving transmission lines with finite conductivity than are frequency domain methods. However, their other advantages are such that many attempts to incorporate frequency dependence into them have been made. The results of operational solutions are often used in conjunction with superposition. Lumped element circuit approximations have been made (section 16.3)) and recently the approach of formulating the equations in the frequency domain and numerically transforming them into the time domain has achieved prominence. This latest approach is perhaps the most successful.

##### 16.4.1) McELROY (1963)

McELROY solves for surge propagation on a multiple circuit line with an earth return. Frequency dependence of the line parameters is taken into account using CARSON's formulation. The line parameters are computed at a single chosen frequency, then, with losses neglected, the resulting T.L.E.s are separated into modal components. In the example given, neglecting losses introduces an error in the velocity of propagation of 0.19%. Having separated the transmission system into an uncoupled component system, the line components are treated as being lossless and the earth component is propagated according to SUNDE'S step responses (section 16.2.3)).

It is worth noting that McELROY'S work on separating the T.L.E.s into an uncoupled modal component system parallels that of WEDEPOHL (1963) published later in the same year where the method is given a more general and extensive treatment.

#### 16.4.2) BICKFORD (1967)

BICKFORD develops a comprehensive power system transients program using the Bewley Lattice method (section 4.4.2)). Frequency dependence of the transmission line parameters is included by separating into modal components and propagating the each component according to the step response of a single circuit transmission line with an earth return. This step response is found by numerically inverting its fourier transform corrected according to CARSON'S (1926) formulation. As in McELROY (1963) the modal transformation is calculated at a single frequency. (The transformation matrices actually have a variation of the order of 5% over the range of frequencies up to 1 MHz.)

The step response is obtained from:

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\gamma(\omega)x}}{j\omega} e^{j\omega t} d\omega$$

To obtain this BICKFORD evaluates the integral:

$$F(t) = \frac{2}{\pi} \int_{-\infty}^{\infty} -\mathcal{I}_m \left\{ \frac{e^{-\gamma(\omega)x}}{j\omega} \right\} \sin(\omega t) d\omega$$

where  $\gamma(\omega) = \sqrt{Z(\omega)Y(\omega)}$  and  $Z(\omega)$  is calculated according to CARSON'S formulation.

16.4.3) BUNDER (1970)

BUNDER models a single circuit transmission line by a two port network (Fig. (16.5)).

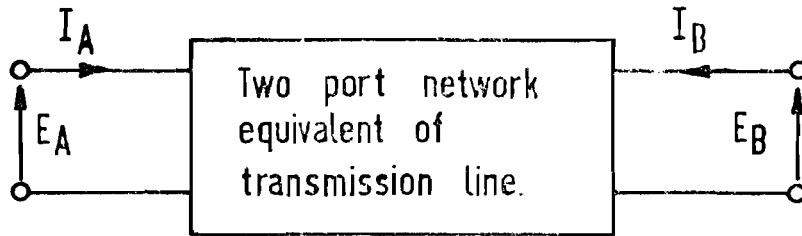


Fig. (16.5)

The network two port equations are written and solved in the frequency domain for a steady state sinusoidal signal. These equations are then numerically inverse fourier transformed giving a set of time domain equations. Arbitrary frequency dependence is simply incorporated before inverse transformation. Specifically BUNDER considers a fully symmetric two circuit transmission line with an earth return. Modal components are solved for. The finite conductivity of the ground is included using CARSON'S (1926) formulation and its effect on each mode is considered.

The two port network equations of the transmission line, Fig. (16.5), can be manipulated into:

$$I_A = Y_{AA} E_A + Y_{AB} E_B$$

$$I_B = Y_{AB} E_A + Y_{BB} E_B$$

where  $Y_{AA} = Y_{BB} = \sqrt{\frac{Y(\omega)}{Z(\omega)}} \coth (\sqrt{Z(\omega)Y(\omega)} \ell)$

$$Y_{AB} = Y_{BA} = \sqrt{\frac{Y(\omega)}{Z(\omega)}} \operatorname{csch} (\sqrt{Z(\omega)Y(\omega)} \ell)$$

These equations are transformed into the time domain giving:

$$i_A(t) = \int_{-\infty}^{\infty} \bar{y}_{AA}(t-T)e_A(T)dT + \int_{-\infty}^{\infty} \bar{y}_{AB}(t-T)e_B(T)dT$$

$$i_B(t) = \int_{-\infty}^{\infty} \bar{y}_{AB}(t-T)e_A(T)dT + \int_{-\infty}^{\infty} \bar{y}_{AA}(t-T)e_B(T)dT$$

The functions  $\bar{y}_{AA}(t)$  and  $\bar{y}_{AB}(t)$  are found by numerical inverse fourier transformation. In example solutions given the convolution integrals were evaluated using Simpson's rule.

#### 16.4.4) SNELSON (1972)

This reference incorporates frequency dependent transmission line parameters into the Method of Characteristics. As with BUNDER (1970) this is achieved first by formulating the characteristic equations in the frequency domain and then numerically inverse fourier transforming them into the time domain. Using the current conventions shown in Fig. (16.6) the characteristic equations for a lossless transmission line are

$$B_m(t) = e_m(t) - Z_C i_m(t) = e_k(t-T) + Z_C i_k(t-T) = F_k(t-T)$$

$$B_k(t) = e_k(t) - Z_C i_k(t) = e_m(t-T) + Z_C i_m(t-T) = F_m(t-T)$$

where  $B(t)$  is defined as 2 × Backward characteristic

$F(t)$  " " " 2 × Forward " " "

$$Z_C = \sqrt{\frac{L}{C}}$$

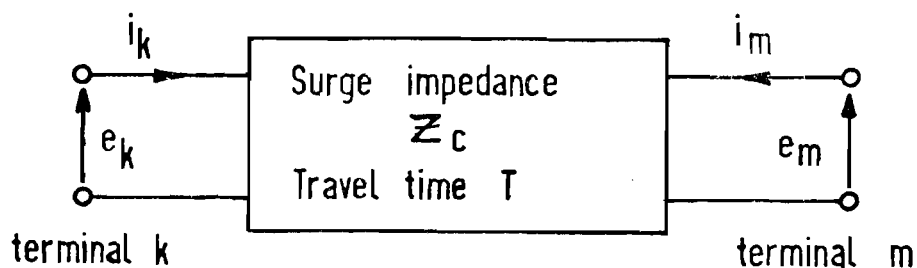


Fig. (16.6) Single circuit transmission line.

The steady state sinusoidal solution of the T.L.E.s for a line of length 'd' is:

$$\bar{e}_k = \bar{e}_m \cosh(\gamma d) - \bar{i}_m Z \sinh(\gamma d)$$

$$\bar{i}_k = -\bar{i}_m \cosh(\gamma d) + \frac{\bar{e}_m}{Z} \sinh(\gamma d)$$

where  $\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$  and  $Z = \sqrt{\frac{(R+j\omega L)}{(G+j\omega C)}}$

With the finite conductivity of the earth included both R and L become functions of frequency and  $\lim_{\omega \rightarrow \infty} Z = Z_c$ .

In the frequency domain:

$$\bar{F}_k = \bar{e}_k + Z_c \bar{i}_k$$

$$\bar{B}_k = \bar{e}_k - Z_c \bar{i}_k$$

with similar expressions for  $\bar{F}_m$  and  $\bar{B}_m$ .

Eliminating the variables  $\bar{e}_k$ ,  $\bar{i}_k$ ,  $\bar{e}_m$  and  $\bar{i}_m$  from the above equations gives the characteristic equations in the frequency domain:

$$\bar{B}_k = \bar{A}_1 \bar{F}_m + \bar{A}_2 \bar{F}_k$$

$$\bar{B}_m = \bar{A}_1 \bar{F}_k + \bar{A}_2 \bar{F}_m$$

$$\text{where } \bar{A}_1 = \frac{1}{\cosh(\gamma d) + \frac{1}{2} \left( \frac{Z_c}{Z} + \frac{Z}{Z_c} \right) \sinh(\gamma d)}$$

$$\bar{A}_2 = \frac{-\frac{1}{2} \left( \frac{Z_c}{Z} - \frac{Z}{Z_c} \right) \sinh(\gamma d)}{\cosh(\gamma d) + \frac{1}{2} \left( \frac{Z_c}{Z} + \frac{Z}{Z_c} \right) \sinh(\gamma d)}$$

Transformed into the time domain these become (cf. equations derived in section 12)):

$$B_k(t) = \int_{-\infty}^{\infty} A_1(u) F_m(t-u) du + \int_{-\infty}^{\infty} A_2(u) F_k(t-u) du$$

$$B_m(t) = \int_{-\infty}^{\infty} A_1(u) F_k(t-u) du + \int_{-\infty}^{\infty} A_2(u) F_m(t-u) du$$

where the inverse fourier transforms required to find  $A_1(t)$  and  $A_2(t)$  are carried out numerically.

#### Comment on BUNDER (1970) and SNELSON (1972)

Both these solution methods permit inclusion of an arbitrary frequency dependence. In both, the calculation of the weighting functions would be a long process but longer for BUNDER than for SNELSON. The two port equations used by BUNDER contain the effects of reflection at each end of the line. Thus the functions  $\bar{y}_{AA}(t)$  and  $\bar{y}_{AB}(t)$  contain many peaks accounting for the signal being reflected back and forth along the line. Evaluation of the convolution integrals would correspondingly involve integrations over a

large range of arguments. In the example given by BUNDER the range of the convolution integrals was  $0 \leq T \leq 79$  milliseconds with the time increment  $\Delta T = 39.1$  micro seconds. By using the method of characteristics and formulating the line propagation functions in terms of both voltage and current, instead of just voltage or current SNELSON removes the problem of reflection and refraction coefficients\*. The weighting functions thus have only one peak and need be evaluated over a comparatively small range of arguments. In an example given only 48 points were used in the convolution integrals. BUNDER'S example used 2048 points.

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\*Footnote: A major advantage of the method of characteristics is that reflection and refraction coefficients are implied by the values of voltage and current at the line boundaries and do not therefore need to be included in the transfer function. This point has not been appreciated by those who discussed this paper or by SNELSON himself. The simplification is, however, clearly demonstrated in section 12) of this thesis where the weighting functions have been derived analytically.

#### 16.4.5) WAGNER (1963)

WAGNER uses the time domain diffusion formulation of MILLER (section 15.3.4)) to solve for the effects of a finitely conducting ground return. He uses the solution for parallel transverse diffusion of current density into the earth resulting from an applied step of current,  $I_x$ .



$$J = \frac{Ix}{s\sqrt{\pi t}} e^{-\frac{x^2}{4s^2 t}}$$

$$\text{where } s = \sqrt{\frac{\rho \times 10^9}{4\pi}} \text{ cm}/\sqrt{\text{sec}}$$

From this an instantaneous value of ground resistance power dissipation is obtained:

$$W(t) = \sqrt{20} \times 10^{-5} \sqrt{\frac{\rho}{t}} \cdot I_x^2$$

from which the average loss up to any instant  $t$  is obtained.

$$W_{AV}(t) = \frac{1}{t} \int_0^t W(t) dt = \sqrt{80} \times 10^{-5} \sqrt{\frac{\rho}{t}} \cdot I_x^2$$

These equations have been obtained for a uniformly spread initial impulse of current density on the surface of the earth. However, the actual initial charge distribution will have a lateral profile (Fig. (16.7)) which when translated into current flowing on the surface of the earth gives:

$$J = \frac{1}{\pi} \frac{h}{h^2 + y^2} I_x \text{ amps/lateral cm.}$$

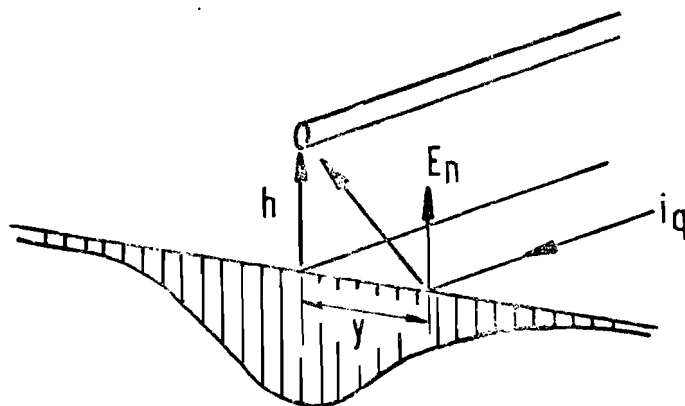


Fig. (16.7)

WAGNER then determines the average loss up to any instant  $t$  for the lateral profile initial conditions as:

$$\begin{aligned} W_{AV}' &= \int_{-\infty}^{\infty} W_{AV} J^2 dy \\ &= \frac{\sqrt{20} \times 10^{-5}}{\pi} \cdot \frac{1}{h} \sqrt{\frac{\rho}{t}} \cdot I_x^2 \quad \text{watts/cm} \end{aligned}$$

He then computes the average energy loss over the entire length of the wave:

$$\begin{aligned} W_{AV}'' &= \frac{1}{t} \int_0^t W_{AV}' dt \\ &= \frac{\sqrt{80}}{\pi} \times 10^{-5} \frac{1}{h} \sqrt{\frac{\rho}{t}} I_x^2 \quad \text{watts/cm} \end{aligned}$$

For a given time an effective resistance can be computed from  $W_{AV}''$ . This is then used by WAGNER in CARSON'S solution for a unit step of voltage applied to a constant parameter line (section 4.7.1)).

#### Comment

This analysis is inadequate in a number of respects. Frequency dependence of the inductance is neglected. A fundamental error is introduced by using the uniform one dimensional diffusion equation. This equation cannot be used to obtain the resultant current density distribution by assuming the lateral profile to be composed of infinitely thin axial filaments of current density. To do this the diffusion equation must be solved for initial axial filaments

of current density along the surface of the ground\*. The solution computed is incorrect for the toe of the wave (from many previous examples this is known to be  $\text{erfc}(\infty) = 0$ ) and the method cannot be correctly used for the rest of the wave.

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\*Footnote: For future reference this problem was solved for the case of 'n' filaments of current density initially on the surface of a homogeneous ground (Fig. (16.8)).

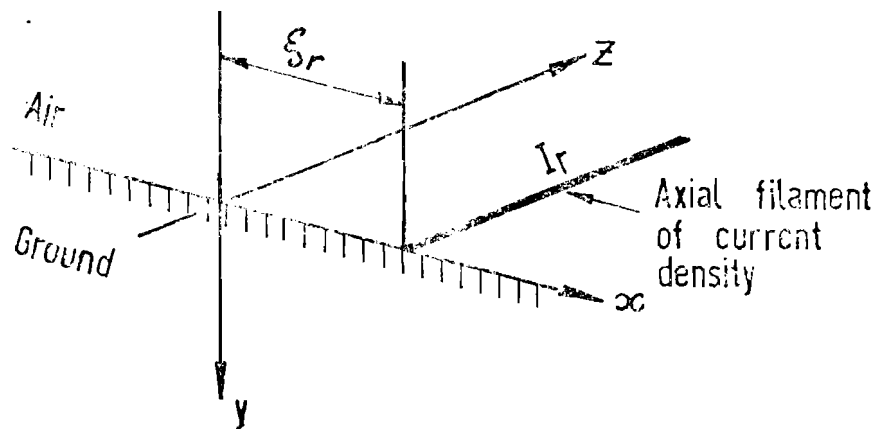


Fig. (16.8) Axial filament of current density on the surface of the ground

The distribution of current density in the ground was obtained as:

$$J(x, y, t) = \sum_{r=1}^n I_r \frac{\mu\sigma}{2\pi t} e^{-\frac{\mu\sigma}{4t}(y^2 + (x - \xi_r)^2)}$$

$$\text{where } J_{r0} = J(\xi_r, 0, 0) = I_r \delta(x - \xi_r) \delta(y)$$

Using the above current density distribution will give the correct results for  $W_{AV}^I$  and  $W_{AV}^{II}$ .

## 16.5) DISCUSSION AND CONCLUSION

Many different approaches have been adopted to solve the problem of propagation along transmission systems with frequency dependent parameters. For steady state sinusoidal signals solution is straightforward. For transient signals the problem becomes more difficult.

Traditionally the approach to transient problems has been to solve for the step response of a line using operational solutions. Usually these are only valid for short or long intervals of time after the wavefront. However ROBERT and TRAN (1969) have developed a step response, corrected according to an operational equivalent of CARSON'S (1926) formulation, which is valid from the wavefront through to moderately large values of time.

Methods involving the numerical evaluation of inverse fourier transforms became popular about 1969 and form perhaps the most successful approach to the problem. These methods can usually accept an arbitrary frequency dependence in the line parameters. Those that solve in the frequency domain are capable of exactly resolving the problem of mutually coupled multiple circuit transmission lines. Those that solve in the time domain can achieve this only when the resolution into modal components is independent of frequency (section 4.8.2)). Usually time domain solutions accept the errors resulting from resolving into modal components with a constant transformation and assume only the earth component to be affected by frequency dependence. The rationale for

this latter assumption is discussed in the introduction of this section.

It is debatable which of the two frequency domain formulated methods is better. Those that solve in the frequency domain and then transform the result into the time domain (Frequency domain methods), or those that transform the equations into the time domain before solving in the time domain (Time domain methods). Frequency domain methods give better solutions for mutually coupled multiple circuit transmission lines but cannot handle with the ease of time domain methods the many non-linearities likely to occur in a typical power system transient or surge situation.

It is difficult to match lumped circuit element networks across as wide a frequency band as can be done with the above methods. If few network elements are used, solution is easier but more approximate. Using more network elements increases the accuracy but also increases the number of equations to be solved.

Of the time domain methods considered, that of SNELSON (1972) stands out as holding much promise. Of the operational solutions that of ROBERT (1969) is remarkable for the time span over which it applies and of the frequency domain solutions those of BATTISSON (1969) and WEDEPOHL (1969) have been applied with accurate results to multiple circuit power system transmission lines. Each of these solutions requires the use of a digital computer. Digital computers have figured prominently in the solutions considered in this analysis. Indeed, many are not practicable without them.

## APPENDIX (16.1)

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\* Reference not seen by author.

+ An uncited reference.

P A R T    3

REFLECTIONS ON TRANSMISSION LINES

## SECTION 17

REFLECTIONS ON TRANSMISSION LINESIntroduction

The purpose of this section is to review progress towards achieving a comprehensive power system transmission line transients program that can account for all first order propagation effects. A diagrammatical summary of the necessary steps in developing such a program is given in Fig. (17.1). It is becoming increasingly important as power system voltages rise to have this program capability. At current experimental transmission levels (1050 kv) the consequence of switching surges and line faults is such that their effects require accurate evaluation. To achieve this one of the first requirements is a comprehensive representation of a typical power system transmission line. Such a line will usually consist of one or two three phase circuits strung from supporting towers above the surface of the earth. Working towards attaining a good transmission line representation in a simple form has been the whole concern of this thesis. However, as will by now be realized, accurate mathematical representation of such a line poses considerable problems, even when interest is restricted to first order propagation effects.

17.1) FIRST ORDER APPROXIMATION TO PROPAGATION ON POWER  
SYSTEM TRANSMISSION LINES

As in any analysis of an electromagnetic problem the starting point is Maxwell's equations. However, their direct

application to power system transient problems would be almost impossible. The problems encountered are of such extent and complexity that many simplifying assumptions are necessary to reduce their solution to a practical engineering basis.

The most basic description of wave propagation along a transmission line is contained in the T.L.E.s with constant coefficients. This set of equations is a consequence of assuming plane wave propagation and expressing the two space variables mutually at right angles to the direction of propagation in terms of equivalent parameters (inductance, capacitance). In terms of Maxwell's equations this simplification is rigorously valid only for perfect conductors embedded in a perfect dielectric. However, the error will be negligible for efficient transmission lines (CARSON, 1928), (RAYMO, 1965). Efficient transmission is nearly always satisfied by the materials used in constructing transmission lines and so the constant coefficient T.L.E.s form a commonly used first approximation to wave propagation.

Two materials associated with power system transmission lines can, under certain circumstances, destroy the conditions required for the above simplification. These are the earth and the air. The earth, when it becomes part of the conducting circuit introduces a poor conductor of large cross section. The air, under the effect of high electric field intensities, ionizes and ceases to be a perfect dielectric. Both these conditions must be included to account for all first order propagation effects on power system transmission lines.

A comprehensive representation of a power system transmission line should therefore be capable of handling;

- (1) efficient transmission (good conductors and good dielectric),
- (2) poor conductors and good dielectric (fault condition when earth becomes part of the conducting circuit),
- (3) good conductors and poor dielectric (very high line voltages),
- (4) poor conductors and poor dielectric (faulted condition with very high voltages).

These categories are shown numbered as above in Fig. (17.1). The first two parts of this thesis have dealt in detail with the first two of these categories.

#### 17.1.1) Good Conductors and Good Dielectric (Efficient Transmission)

This enables propagation to be represented by the T.L.E.s with constant coefficients. Even in this simple form, difficulties in solving the T.L.E.s are encountered. Exact solutions exist only for:

- 1) Lossless lines (all signals)
- 2) Distortionless lines (all signals)
- 3) Steady state sinusoidal signals (all lines)
- 4) Applied unit step signal - with or without a source resistance (all single circuit lines).

Even then the solution 4) involves integrals that cannot be evaluated analytically in terms of simple mathematical functions.

Mutually coupled multiple circuit transmission lines can be handled exactly only for:

- 5) Dispersionless lines (all signals)
- 6) Fully symmetric lines (all signals)
- 7) Steady state sinusoidal signals (all lines).

The problem of transient propagation on a typical power system transmission line seldom conforms to these exact solution categories and approximate solutions must be developed. Development of approximate solutions suitable for power system transient analyses occupies most of the Linear Transmission Line Analysis given in this thesis. A number of power system transients programs based on the T.L.E.s with constant coefficients have been developed. A recent example is that of DOMMEL (1969). The approximate solutions developed in the Linear Transmission Line Analysis are superior to those commonly used in such programs both in accuracy and/or computational efficiency. Any new transients program could profitably use these results.

The subject area of efficient transmission appears to be in a sound and satisfactory state of development, adequate for most engineering needs where it would be applied.

#### 17.1.2) Poor Conductors and Good Dielectric

Here the T.L.E.s have coefficients that are functions of frequency. For typical power system transmission lines under fault conditions only the series inductance and resistance of the transmission lines are affected. Neither an exact formulation of this frequency dependence nor a solution of the resulting equations has been developed. Nor could such a rigorous solution ever be justified in view of the inaccurate

knowledge of the physical constants involved in any practical situation. The formulations of SUNDE (1968) and WEDEPOHL (1966) are of sufficient accuracy for most transient situations and that of CARSON (1926) for common power system frequencies. These formulations account for all the major effects of the earth's finite conductivity and it is not anticipated that future development will see them significantly improved upon.

Solving the T.L.E.s with frequency dependent coefficients has been an area of much activity over the past six or so years. Most recent developments have involved numerical techniques on digital computers. Good results have been obtained by numerically inverting the fourier transform. Comprehensive solutions of transmission along power system transmission lines have been obtained by frequency domain techniques (BATTISSON (1969) and WEDEPOHL (1969)). A promising time domain technique has been proposed by SNELSON (1972) and an extended step response by ROBERT (1969). Thus there exist a number of promising solution techniques. A comprehensive power system transients program using these solution techniques has not yet been encountered by the author. BICKFORD (1967) has produced an excellent transients program where frequency dependence is included by a step response derived numerically from its fourier transform. However, the methods of BATTISSON and WEDEPOHL are more accurate and that of SNELSON better suited to the power system situation.

It is felt that some improvement can still be made in this subject area.



### 17.1.3) Non-ideal Dielectric and Good Conductors

This situation has not been previously dealt with in this thesis and the subject area has not been as widely read.

For these conditions the air around the conductor ionizes above a certain voltage. This extracts energy from the transmission system which is not recovered when the voltage decreases. Thus propagating signals are very attenuated and distorted. Further, the envelope of conducting ions around the conductor increases the capacitance of the line. However, this has small effect compared with the energy required for ionization (SUNDE, 1968). The classic formulation of the ionization energy derived by PEEK (1929) is empirical and based on experiments with alternating currents. HYLTEN-CAVALLIUS (1959) points to a limited knowledge of corona under impulse conditions which seems to show that corona under impulse conditions differs fundamentally from corona under alternating current conditions. He derives the T.L.E.s with corona included by assuming the charge absorbed by the line and the propagation velocity to be a function of line voltage. SUNDE (1968) uses PEEK'S quadratic voltage formulation as does BEWLEY (1963).

Solving the resulting T.L.E.s is a difficult problem owing to their non-linear nature. More recently the technique of finite differences (sections 4.4.5) and (4.4.7)) has been advanced as a method of obtaining comprehensive solutions for multiple circuit non-linear transmission lines (TALUKDAR, 1972) (RAU, 1972). Neither of these references has considered any except simple non-linearities. It is

quite possible on account of non-linear capacitive effects that problems of numerical stability will result. Under these circumstances the method of characteristics being numerically stable may offer a viable alternative although its application would become iterative. No comprehensive power system transients programs based on corona formulated non-linear T.L.E.s have been encountered by the author.

Although the author's reading in this area is admittedly limited it does appear to be an area that would benefit from further investigation.

#### 17.1.4) Poor Conductors and Poor Dielectric

Here the propagating wave is affected by two factors simultaneously, one of which is non-linear. The resulting distortion cannot be determined as the algebraic sum of each of the distorting factors. HYLTEN-CAVALLIUS (1959) illustrates this by solving for a signal propagating along a line composed of two parts in series. In one part the signal is subjected only to the effects of corona. In the other part it is subjected only to the effects of the earth's finite conductivity. Reversing the order of propagation gives totally different results. Assuming that the truth will fall somewhere between the extreme cases it can be stated that the resulting distortion will be smaller than the algebraic sum of the two factors but larger than for the least effective single factor. HYLTEN-CAVALLIUS expresses the above statement by a sum of squares relationship. However, this will at best give only a rough indication of the true result.

An added difficulty in this situation is that the solution methods most suited to frequency dependence are the least suited to corona and vice versa.

This category represents a faulted condition on an E.H.V. transmission line. It is a transmission line transient likely to occur on power systems of the future. Both in formulating the distorting effects and solving the resulting equations much work has yet to be done before an adequate description of propagation is obtained.

## 17.2) CONCLUSION

A comprehensive power system transmission line transients program that can account for all first order propagation effects is still far away. The first approximation of efficient linear transmission is well established. It is in a more or less complete state of development and satisfactory for application to many engineering problems. Propagation with finite conductivity (frequency dependence) is also in an advanced state of analysis. The major effects have been formulated and a number of promising solution procedures have been produced. Comprehensive transients programs including both the above have been produced. Surge propagation under conditions of air ionization (corona) appears to be not so advanced. Both formulation of the effects and solution of the resulting equations appear to be candidates for further development. Little seems to have been attempted on the all-inclusive propagation category where both frequency dependence and corona are simultaneously

present. It may even be that this category will require further formulation as separate from each of the individual component effects. Until propagation in the presence of both frequency dependence and corona has been satisfactorily resolved, a power system transmission line representation that includes all first order propagation effects will not have been obtained.

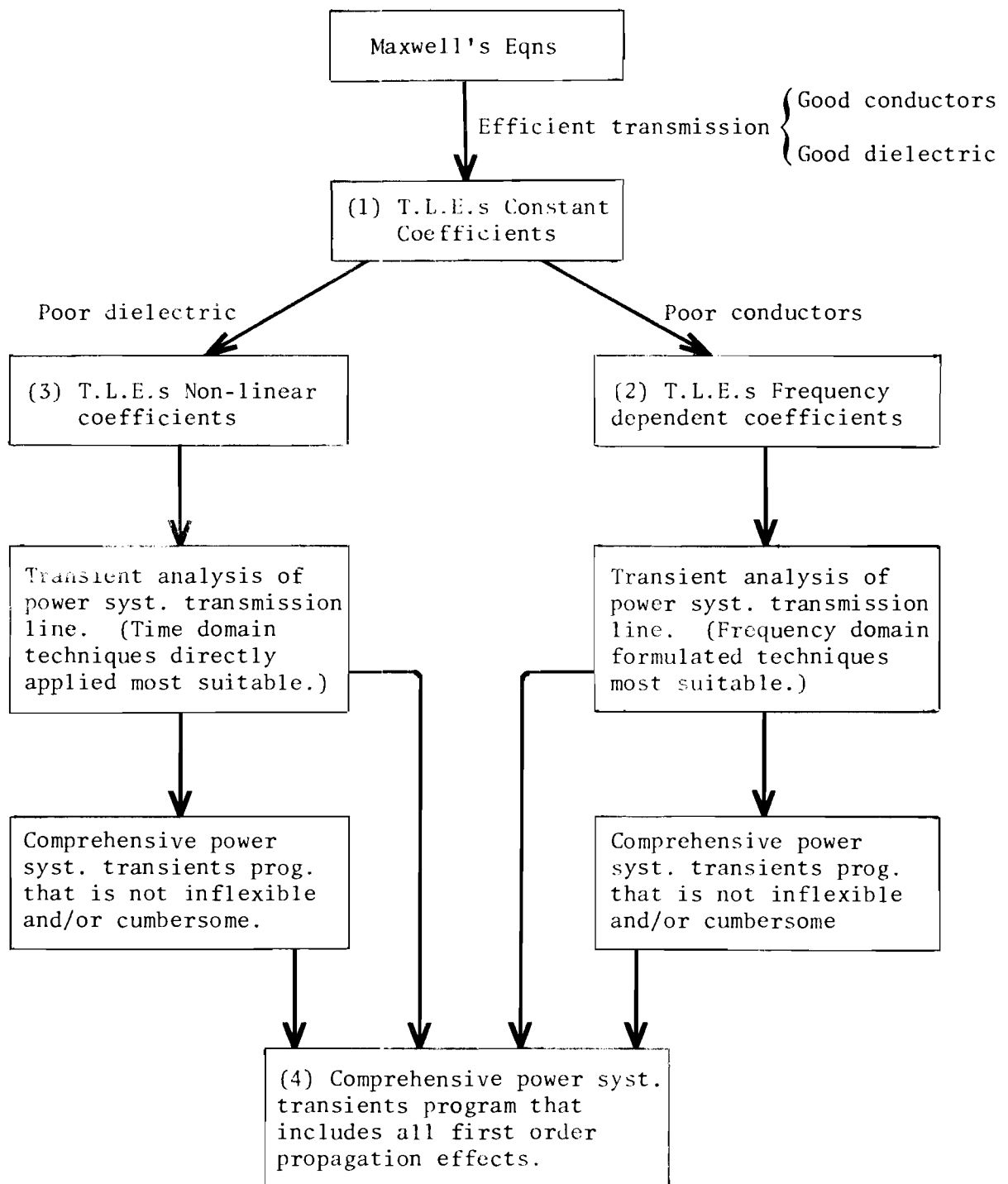


Fig. (17.1) Diagrammatical summary of steps in achieving a comprehensive power system transmission line transients program that can account for all first order propagation effects.

## APPENDIX (17.1)

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